A noncausal strategy for feedback control of spatially–parabolic flow systems

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***Abstract*— We present a noncausal framework for model-based feedback stabilization of a large class of spatially-developing boundary-layer flow systems. The systems considered are (approximately) parabolic in the spatial coordinate x. This facilitates the application of a range of established feedback control theories which are based on the solution of differential Riccati equations which march over a finite horizon in x (rather than marching in t, as customary). However, unlike systems which are parabolic in time, there is no causality constraint for the feedback control of systems which are parabolic in space; that is, downstream information may be used to update the controls upstream. Thus, a particular actuator may be used to neutralize the effects of a disturbance which actually enters the system downstream of the actuator location. In the present study, a feedback control strategy is proposed which takes advantage of this special capability of feedback control rules in the spatially-parabolic setting in order to minimize a globally-defined cost function in an effort to maintain the flow laminar. Numerical results which verify the effectiveness of this approach is presented.**

I. INTRODUCTION

The present paper develops a closed-loop, Riccati-based *feedback* control strategy (as opposed to an open-loop, adjoint-based control *optimization* strategy) to stabilize a spatially-developing boundary layer flow system. Control is applied via a blowing/suction distribution over a portion of the wall, and state estimation is accomplished via measurements of skin friction and pressure distributed over the same region (for a dicussion on the actual implementation of such a distributed control see Bewley [4]). The purpose of the control in this problem is to keep the flow perturbations sufficiently small that transition to turbulence is inhibited.

An important characteristic of laminar[1] systems of this type is that they are essentially independent of time, and the equations that govern them, subject to the correct approximations, are *parabolic* in the spatial coordinate x.
Hill [11] pointed out the role of adjoint systems in the local receptivity problem for boundary-layer flow systems. By using an iterative adjoint-based optimization strategy Luchini [13] and Andersson et al. [2] found the optimal perturbations of the boundary-layer flow. Iterative (adjoint-based) control optimization strategies for boundary-layer flow systems are appropriate for open-loop control optimizations, and are beginning to see successful applications in this regard. For recent reviews of this line of research, see, e.g., Walter et al. [20], Cathalifaud & Luchini [7], and Pralits et al. [16]. However, it is computationally quite difficult (if not impossible) to apply iterative, adjoint-based control optimization strategies in the closed-loop setting to neutralize the effects of the random flow disturbances that arise in nature. For such problems, feedback control strategies which can respond quickly and in a coordinated fashion to measurements of the flow system are necessary.

Control strategies for systems which evolve parabolically in time must be *causal*; that is, they must depend only on present and past measurements of the flow. However, control strategies for systems which evolve parabolically in space are not limited by such a constraint; the control at a particular actuator location may depend on measurements taken both upstream and downstream. Thus, to exploit the additional measurement information available in this setting, a different set of tools is needed for this problem beyond the standard LQG ($\mathcal{H}_2$) framework and "robustifying" extensions thereof ($\mathcal{H}_\infty$, LTR, etc.).

II. GOVERNING EQUATIONS

Based on the dimensional coordinates $\{x^*, y^*, z^*\}$, velocities $\{u^*, v^*, w^*\}$, and pressure p^*, we define the dimensionless quantities $x = x^*/L$, $\{y, z\} = \{y^*, z^*\}/\delta$, $u = u^*/U_o$, $\{v, w\} = \{v^*, w^*\}Re_\delta/U_o$, and $p = p^*\,Re_\delta^2/(\rho U_o^2)$, where U_o is the freestream velocity, ρ is the density, μ is the viscosity, $\nu = \mu/\rho$ is the kinematic viscosity, L is a reference streamwise length, $\delta = \sqrt{L\nu/U_o}$ is a reference boundary layer thickness, and $Re_\delta = U_o\,\delta/\nu$ is a reference Reynolds number. Also, from the dimensional radius of curvature r^* of the surface in the x-y plane, we define the dimensionless curvature parameter $\epsilon = \delta/|r^*|$, the Görtler number $G = Re_\delta\sqrt{\epsilon}$, and a sign function s such that $s = 0$ corresponds to a flat wall, $s = 1$ corresponds to a concave wall, and $s = -1$ corresponds to a convex wall. In order to apply the

[1]A laminar flow is characterized by the regularity of the trajectories of the fluid particles, producing contiguous layers of fluid which slip the ones on the others and do not mix.

boundary-layer approximation[2] and to develop a linear set of equations governing small perturbations to the nominal (undisturbed) boundary-layer flow, we assume that $\delta \ll L$ (*i.e.*, $Re_\delta \gg 1$), $\delta \ll |r^*|$ (*i.e.*, $\epsilon \ll 1$), $G \lesssim O(1)$, and that the nominal flow $\mathbf{Q} = (U, V, W, P)^T$ is laminar, steady and spanwise–invariant.

Small three-dimensional perturbations $\mathbf{q} = (u, v, w, p)^T$ to the nominal flow $\mathbf{Q} = (U, V, W, P)^T$ are governed by the linearized Navier-Stokes equation. As the system governing these perturbations is linear and homogeneous in z, we may decouple the various spanwise modes of this system by taking the Fourier transform of all perturbation variables with spanwise variation (namely, the state, the controls, the measurements, and the disturbances) in the z direction (see Bewley & Liu [5]). In the present discussion, we therefore consider a particular Fourier mode of the flow perturbations, and assign a variation in z of $\exp(-i\beta z)$ to all of these variables. Once the control problem is solved for a series of spanwise wavenumbers, inverse Fourier transform of the feedback gains should lead to feedback convolution kernels which are spatially localized in the spanwise coordinate (see, e.g., Bamieh et al. [3] and Bewley [4]).
Following the analysis of Hall [10], under the boundary-layer assumptions itemized above, the linearized, nondimensional equations for the flow perturbations reduce to

$$\begin{aligned} &(Uu)_x + Vu_y + U_y v - i\beta W u - u_{yy} + \beta^2 u = 0,\\ &Uv_x + V_x u + (Vv)_y + p_y - i\beta W v + 2sG^2 Uu - v_{yy} + \beta^2 v = 0,\\ &Uw_x + W_x u + V w_y + W_y v - i\beta p - i\beta W w - w_{yy} + \beta^2 w = 0,\\ &u_x + v_y - i\beta w = 0, \end{aligned} \tag{1}$$

where a subscript means a differential operation, and the boundary and initial conditions are:

$$\begin{aligned} &u = w = 0,\ v = v_w(x),\ \text{at } y = 0\\ &u = v = w = 0,\ \text{at } y = \infty\\ &\{u, v, w\} = \{u_0, v_0, w_0\},\ \text{at } x = x_0, \end{aligned} \tag{2}$$

where $v_w(x)$ is the control velocity of blowing/suction distributed over the wall on the strip $x_0 < x < L$.
Define the normal dimensionless vorticity $\eta = -i\beta u$, we now combine the governing equations (1) in such a way as to determine a set of two coupled equations for the perturbation components of the normal velocity and normal vorticity. Defining $D^k = \partial^k/\partial y^k$ the result is

$$\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & \tilde{E}_{22} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}, \tag{3}$$

where: $\tilde{E}_{11} = U(D^2 - \beta^2) - U_{yy}$, $\tilde{E}_{12} = -(2i/\beta)[U_{xy} + U_x D^1]$, $\tilde{E}_{22} = -U$, $\tilde{A}_{21} = -i\beta U_y$, $\tilde{A}_{11} = -\Big[(V_{yy} - \beta^2 V)D^1 + V_{yyy} + V_y(D^2 - \beta^2) + VD^3 - D^4 + 2\beta^2 D^2 - \beta^4 + i\beta W_{yy} - i\beta W D^2 + i\beta^3 W\Big]$, $\tilde{A}_{12} = -i/\beta\Big[V_{xyy} - V_x(D^2 + \beta^2) + 2i\beta(W_x D^1 - W_{xy}) - 2\beta^2 G^2 U\Big]$, and $\tilde{A}_{22} = \Big[U_x + VD^1 - D^2 + \beta^2 - i\beta W\Big]$.

[2] The boundary layer assumption is that the boundary-layer thickness is much smaller than the streamwise length scales in the system, and that the time scale of the external perturbations to the system are large with respect to the boundary-layer thickness divided by the freestream velocity (see, e.g., [17]).

III. FORMULATION OF THE DISCRETIZED SYSTEM

We now perform a discretization of the system in the y coordinate on a finite number of discretization points with the appropriate grid stretching. Let $\{\mathbf{v}, \boldsymbol{\eta}\}$ denote the spatial discretizations of $\{v, \eta\}$ on the interior of the domain. Define the matrices $\{\hat{E}_{11}, \hat{E}_{12}, \hat{E}_{22}, \hat{A}_{11}, \hat{A}_{12}, \hat{A}_{21}, \hat{A}_{22}\}$ as the spatial discretizations of $\{\tilde{E}_{11}, \tilde{E}_{12}, \tilde{E}_{22}, \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22}\}$ on the interior of the domain using the chosen technique, and the vectors $\mathbf{e}_{11}$ and $\mathbf{a}_{11}$ to denote the influence of the normal velocity at the wall on, respectively, the left-hand side and right-hand side of the v component of the discretization of (3). Using these discrete forms, it is straightforward to express (3) in the state-space form

$$\mathbf{q}_x = A\mathbf{q} + B\,\phi, \quad \text{where} \mathbf{q} = \begin{pmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\eta} \end{pmatrix} \\ v_w \end{pmatrix}, \tag{4}$$

and $A = \begin{pmatrix} \hat{E}^{-1}\hat{A} & \hat{E}^{-1}\mathbf{a} \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -\hat{E}^{-1}\mathbf{e} \\ 1 \end{pmatrix}$,

$$\hat{E} = \begin{pmatrix} \hat{E}_{11} & \hat{E}_{12} \\ 0 & \hat{E}_{22} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix},$$

$$\mathbf{e} = \begin{pmatrix} \mathbf{e}_{11} \\ 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \mathbf{a}_{11} \\ 0 \end{pmatrix}.$$

The control variable in this formulation is $\phi = dv_w/dx$.
To account for external system disturbances and modeling uncertainties, we now modify the state equation (4) by adding disturbances $\mathbf{w}$ to the right-hand side:

$$\mathbf{q}_x = A\mathbf{q} + B\phi + D\mathbf{w}, \tag{5}$$

where the disturbance vector $\mathbf{w}$ depends on the spatial coordinate x. The matrix D in (5), which represents the square root of the external disturbances covariance, reflects which disturbances affect the most the system. We desire to develop a global strategy in which the control $\phi(x)$ may actually respond to disturbances $\mathbf{w}(x)$ acting over the entire domain under consideration $x_0 \le x \le L$. To facilitate this in the standard (causal) setting, we first discretize the system in x, where $x_k = x_0 + k\Delta$, $k = 0, \ldots, N$, and $\Delta = (L - x_0)/N$ represents the grid spacing in x. Then we define an augmented state

$$\mathbf{q}_k^a = \begin{pmatrix} \mathbf{q}_k \\ \mathbf{q}_k^{\mathbf{w}} \end{pmatrix},$$

$$\text{where} \quad \mathbf{q}_0^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \end{pmatrix}, \quad \mathbf{q}_1^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \\ 0 \end{pmatrix}, \ \dots \ \mathbf{q}_N^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_N \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

Note that the augmented state $\mathbf{q}_k^a$ at a particular streamwise station x_k only includes the disturbances entering the system downstream of that location, as the influence of the disturbances upstream are accounted for in $\mathbf{q}_k$. Note also that we can express the evolution of $\mathbf{q}_k^{\mathbf{w}}$ in the discrete state-space form

$$\mathbf{q}_{k+1}^{\mathbf{w}} = A^d \mathbf{q}_k^{\mathbf{w}}, \qquad A^d = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}, \tag{7}$$

where the relation between $\mathbf{w}_k$ and $\mathbf{q}_k^{\mathbf{w}}$ is

$$\mathbf{w}_k = M^{\mathbf{w}} \mathbf{q}_k^{\mathbf{w}}, \quad M^{\mathbf{w}} = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}. \tag{8}$$

By combining equations (5), (7), and (8), we can obtain a state-space formulation for the augmented state $\mathbf{q}^a$. However, the inherently discrete nature of the evolution of our disturbance model $\mathbf{q}^{\mathbf{w}}$ compels us to first derive a discrete–in–x formulation of the state equation (5). To accomplish this, we approximate $\{A, B, \mathbf{q}, \phi\}$ with $\{A_k, B_k, \mathbf{q}_k, \phi_k\}$ over the interval $x_k \le x < x_{k+1}$ for each value of k, where, e.g., $A_k = A(x_k)$. Using this approximation (comonly referred to as a "zero-order hold"), we may express (5) in the following "delta form" [15]:

$$\delta \mathbf{q}_k = \Omega_k A_k \mathbf{q}_k + \Omega_k B_k \phi_k + \Omega_k D_k \mathbf{w}_k, \tag{9}$$

where $\Omega_k = (1/\Delta) \int_0^\Delta \exp(A_k \tau)\, d\tau$ and $\delta \mathbf{q}_k = (\mathbf{q}_{k+1} - \mathbf{q}_k)/\Delta$. Note in particular that $\Omega_k \to I$ as $\Delta \to 0$, and thus the discrete-in-x relation (9) tends towards the continuous-in-x relation (5) as the grid is refined. This behaviour of the δ-formulation also follows for the Riccati and Lyapunov equations that arise in the control and estimation problems in the following sections, and is an appealing characteristic of this particular discrete formulation.
Combining (9), (7), and (8), we finally obtain a discrete, causal state-space formulation for the augmented state, to which standard control theories may be applied:

$$\begin{aligned} &\delta \mathbf{q}_k^a = A_k^a \mathbf{q}_k^a + B_k^a \phi_k, \\ &\text{where} \quad A_k^a = \begin{pmatrix} \Omega_k A_k & \Omega_k D_k M^{\mathbf{w}} \\ 0 & A^d \end{pmatrix} \quad \text{and} \quad B_k^a = \begin{pmatrix} \Omega_k B_k \\ 0 \end{pmatrix}. \end{aligned} \tag{10}$$

IV. OPTIMAL CONTROL FOR NONCAUSAL SYSTEMS

In the original PDE setting, our control objective may be written as finding a feedback control rule which minimizes the cost function

$$\begin{aligned} &\mathcal{J} = \sum_{i=0}^{N} \Delta \left[(\mathbf{q}^a)_i^* Q^a \mathbf{q}_i^a + \alpha_4^2 \, \phi_i^* \phi_i \right], \\ &\text{where} \quad Q^a = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \alpha_1^2 \, I_s & 0 & 0 \\ 0 & \alpha_2^2 \, I_s & 0 \\ 0 & 0 & \alpha_3^2 \end{pmatrix}, \end{aligned} \tag{11}$$

and I_s is a diagonal matrix with the corresponding local grid spacing on the elements of the diagonal. $\mathcal{J}$ represents the sum of the control energy and the perturbation energy over $[x_0, x_N]$.
Note that the technique of augmenting the initial state with the disturbances entering the entire system facilitated the conversion of the noncausal problem described in the introduction into the causal problem represented by (10). Together with the control objective (11), a feedback control rule of the form

$$\phi_k = -K_{k+1} \mathbf{q}_k^a \tag{12}$$

may be found directly using standard "discrete-time" optimal control theory. In fact, as discussed in Bitmead et al. [6], the Riccati equation associated with this control problem may be partitioned in a convenient fashion by defining

$$\begin{aligned} K_k &= \left(\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k \right)^{-1} B_k^* \Omega_k^* \begin{pmatrix} K_k^1 & K_k^2 \end{pmatrix}, \\ K_k^1 &= \Sigma_k^{11} \left(I + \Delta \Omega_k A_k \right), \\ K_k^2 &= \Delta \Sigma_k^{11} \Omega_k D_k M^{\mathbf{w}} + \Sigma_k^{12} \left(I + \Delta A_k^d \right), \end{aligned} \tag{13}$$

where Σ_k^{11} and Σ_k^{12} solve the Riccati and Lyapunov equations

$$\begin{aligned} \overline{\delta} \Sigma_k^{11} &= Q + A_k^* \Omega_k^* \Sigma_k^{11} + \Sigma_k^{11} \Omega_k A_k + \Delta A_k^* \Omega_k^* \Sigma_k^{11} \Omega_k A_k \\ &\quad - (K_k^1)^* \Omega_k B_k \left[\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k \right]^{-1} B_k^* \Omega_k^* K_k^1, \\ \overline{\delta} \Sigma_k^{12} &= A_k^* \Omega_k^* \Sigma_k^{12} + \left[I + \Delta A_k^* \Omega_k^* \right] \left[\Sigma_k^{11} \Omega_k D_k M^{\mathbf{w}} + \Sigma_k^{12} A^d \right] \\ &\quad - (K_k^1)^* \Omega_k B_k \left[\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k \right]^{-1} B_k^* \Omega_k^* K_k^2, \end{aligned} \tag{14}$$

where $\overline{\delta} \Sigma_k = -(\Sigma_k - \Sigma_{k-1})/\Delta$. As $\Delta \to 0$, equations (14) tend towards the corresponding continuous Riccati and Lyapunov equations (cf. Middleton & Goodwin, [15]).
Finally, by combining (12) and (10), we can express ϕ_k as a simple function of the initial augmented state vector $\mathbf{q}_0^a$:

$$\phi_k = K_{k+1}^0 \, \mathbf{q}_0^a, \quad \text{where} \quad K_{k+1}^0 = -K_{k+1} \prod_{i=0}^{k-1} (A_i^a - B_i^a K_{i+1}). \tag{15}$$

V. OPTIMAL ESTIMATION/SMOOTHING

By (15), we see that we can express the optimal control distribution on $x_0 < x < L$ which minimizes the globally-defined cost function J as a simple function of the upstream flow perturbation $\mathbf{q}_0$ and the system disturbances $\mathbf{w}(x)$ between x_0 and L. The task which remains is to find a simple way to obtain a good estimate of $\mathbf{q}_0^a$ based on the available measurements at the wall.

Defining the vector $\boldsymbol{\mu}$ as the measurement noise, the measurements of the streamwise and spanwise skin friction and pressure distributions over the wall may be written as

$$\mathbf{y}(x) = \mathbf{m}(x) + \boldsymbol{\mu}, \qquad \text{where} \quad \mathbf{m}(x) = \left(\frac{\partial u}{\partial y}\bigg|_{wall}, \ \frac{\partial w}{\partial y}\bigg|_{wall}, \ p|_{wall} \right)^T \tag{16}$$

Note that applying the nondimensionalization discussed previously to the definition of η, to the continuity equation, and to the wall-normal momentum equation, and neglecting the terms in $1/Re_\delta^2$, we can express the skin friction and pressure at the wall as:

$$\mathbf{m} = Z\,\mathbf{q} + N\,\phi, \quad \text{where} \quad N = \begin{pmatrix} 0 \\ 0 \\ 1/\beta^2 U_w^{(1)} \end{pmatrix} \quad \text{and} \tag{17}$$

$$Z = \begin{pmatrix} 0 & \frac{i}{\beta}\,\frac{\delta^1}{\delta y^1}\Big|_w & 0 \\ -\frac{i}{\beta}\,\frac{\delta^2}{\delta y^2}\Big|_w^{int} & Z_{22} & \frac{iU_w^{(3)}}{2\beta U_w^{(1)}} - \frac{i}{\beta}\,\frac{\delta^2}{\delta y^2}\Big|_w^w \\ \frac{1}{\beta^2}\,\frac{\delta^3}{\delta y^3}\Big|_w^{int} & 0 & \frac{1}{\beta^2}\,\frac{\delta^3}{\delta y^3}\Big|_w^w \end{pmatrix},$$

where $Z_{22} = \frac{1}{2\beta^2 U_w^{(1)}} \left[-\frac{\partial U_w^{(1)}}{\partial x}\,\frac{\delta^1}{\delta y^1}\Big|_w + \frac{\delta^4}{\delta y^4}\Big|_w - \beta^2\,\frac{\delta^2}{\delta y^2}\Big|_w \right]$, and the notation $\delta^k/\delta y^k\big|_w$ denotes the discretization of the k'th derivative operator evaluated at the wall (the superscript int denotes the influence on the interior of the domain, and w the influence at the wall) and $U_w^{(k)}$ the k'th y-derivative of U evaluated at the wall.

Using the relations (16) and (17), we can approximate the vector of the wall measurements $\mathbf{y}$ as a function of the discrete state vector $\mathbf{q}$, the control variable ϕ, and the measurement noise $\boldsymbol{\mu}$:

$$\mathbf{y} = Z\mathbf{q} + N\phi + \boldsymbol{\mu} \tag{18}$$

Applying the definition of the augmented state $\mathbf{q}^a$, we may write (18) as

$$\mathbf{y}_k = Z^a \mathbf{q}_k^a + N_k \phi_k + \boldsymbol{\mu}_k \tag{19}$$

where $Z^a = (Z \quad 0)$. We now define the notation $\hat{\mathbf{q}}_{k|m}^a = \hat{\mathbf{q}}^a(x_k|x_m)$ to denote the estimate of $\mathbf{q}^a(x_k)$ based on the measurements $\mathbf{y}(x)$ from $x_0 \leq x \leq x_m$. Our aim is to calculate an estimate of $\mathbf{q}_0^a$ based on the measurements $\mathbf{y}(x)$ for $x_0 \leq x \leq x_N = L$ (i.e. $\hat{\mathbf{q}}_{0|N}^a$). This is a "smoothing" problem, and, given the correct manipulations, can be solved using a Kalman filter. To solve this problem, we first substitute the value of ϕ_k obtained in (12) into the equations (10) and (19). Defining $F_k = A_k^a - B_k^a K_{k+1}$ and $H_k = Z_k^a - N_k K_{k+1}$, we have

$$\begin{aligned} \delta \mathbf{q}_k^a &= F_k \mathbf{q}_k^a, \\ \mathbf{y}_k &= H_k \mathbf{q}_k^a + \boldsymbol{\mu}_k. \end{aligned} \tag{20}$$

Defining $\hat{\mathbf{q}}_{0|-1}^a = E(\mathbf{q}_0^a)$, the a priori estimate of $\mathbf{q}_0^a$, and applying Kalman filter theory to the system (20), we obtain the following evolution equation for the estimate $\hat{\mathbf{q}}_{k|k-1}^a$

$$\begin{aligned} \delta \hat{\mathbf{q}}_{k|k-1}^a &= F_k \hat{\mathbf{q}}_{k|k-1}^a + L_k \left[\mathbf{y}_k - H_k \hat{\mathbf{q}}_{k|k-1}^a \right], \\ & \qquad k = 0, 1, 2, \ldots N, \\ L_k &= (\Delta F_k + I)\, P_k H_k^* \left[\Delta H_k P_k H_k^* + C_\mu \right]^{-1} \end{aligned} \tag{21}$$

where P_k is solution of the Riccati equation

$$\delta P_k = P_k F_k^* + F_k P_k + \Delta F_k P_k F_k^* - L_k \left[\Delta H_k P_k H_k^* + C_\mu \right] L_k^*, \qquad k = 0, 1, \ldots N, \tag{22}$$

where P_0 is an estimate of the covariance of the state $\mathbf{q}_k^a$ at $k = 0$ and C_μ is an estimate of the covariance of the noise $\boldsymbol{\mu}$; in practice, P_0 and C_μ are used as design parameters when developing the estimator.Our problem actually differs slightly from the filtering problem (21). In particular, the information we want to reconstruct, $\mathbf{q}_0^a$, must be obtained from measurements taken on $x_0 \leq x \leq x_N$. In other words, we seek to determine the value of $\hat{\mathbf{q}}_{0|N}^a$, not the value of $\hat{\mathbf{q}}_{N+1|N}^a$ which can be obtained from (21). As in Anderson and Moore [1], $\hat{\mathbf{q}}_{0|N}^a$ can be easily derived from the filter problem presented above by marching the discrete equation

$$\hat{\mathbf{q}}_{0|k}^a = \hat{\mathbf{q}}_{0|k-1}^a + \Delta R_k H_k^* \left[\Delta H_k P_k H_k^* + C_\mu \right]^{-1} \left[\mathbf{y}_k - H_k \hat{\mathbf{q}}_{k|k-1}^a \right], \qquad k = 0, 1, 2, \ldots N,$$

where R_k satisfies the Lyapunov equation

$$\delta R_k = R_k (F_k - L_k H_k)^*, \quad k = 0, 1, 2, \ldots N,$$

where $R_0 = P_0$; note that $\hat{\mathbf{q}}_{0|-1}^a = E(\mathbf{q}_0^a)$ as stated previously.

Assuming that the initial state $\mathbf{q}_0$ is a random variable uncorrelated with the disturbances $\boldsymbol{\mu}_k$, it is straightforward to partition this estimation problem like we did previously with the control problem.

We thus obtain $\hat{\mathbf{q}}_{0|N}^a$, which is the best approximation possible of the initial augmented state $\mathbf{q}_0^a$ given all of the measured data on $x_0 \leq x \leq L$. This estimate of the augmented state at x_0 may then be combined with the control relationship (15) to determine the optimal control based on the available noisy measurements.

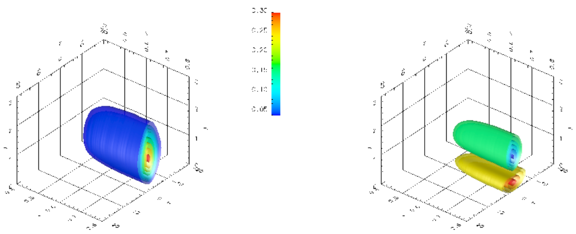

Fig. 1. Isosurfaces of $K^0(x,y,z)$ (left) relating the streamwise component of the velocity $u(x = 0.5, y, z)$, and (right) relating the wall–normal component of the velocity $v(x = 0.5, y, z)$ to the control input $\phi(x, z = 0)$. Note the control domain extension in x = $[0.5, 0.8]$.

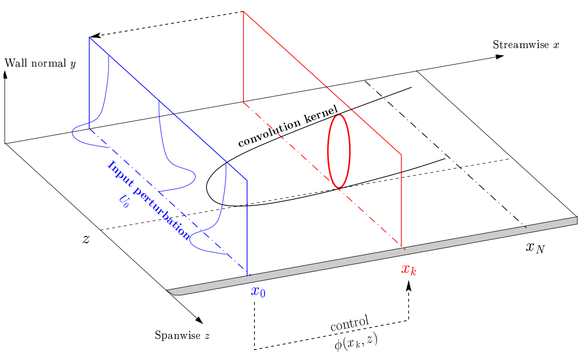

Fig. 2. Relation between the convolution control kernel and the control. The control $\phi(x = x_k, z)$ is found by convolving the convolution kernel K^0 in the plane $x = x_k$ with the augmented state $\mathbf{q}_0^a$ in the vicinity of the spanwise location z.

VI. LOCALIZED CONVOLUTION KERNELS

By inverse Fourier transforming the gain matrices K^0, we obtain convolution control kernels which are spatially localized in the spanwise direction z (see [3] and [12]). Physically, this means that the control at a spanwise location z depends only on the input perturbation in the vicinity of this spanwise location. Figure 1 depict representative convolution kernels relating the streamwise and wall–normal velocity of perturbation at $x_0 = 0.5$ to the control input on the wall as a result of the present control formulation. To obtain the control at the wall position x_k and z, we simply convolve the kernel in the plane at the streamwise location x_k with the input perturbation $\mathbf{q}_0^a$ in the vicinity of the spanwise location z, as depicted in Figure 2. As expected, the convolution kernels depicted in Figure 1 do not exhibit spatial localization in the streamwise direction x, but are elongated in this direction.

It is significant to note that our objective function, which up to this point has been to minimize the perturbation energy over the entire streamwise extent $[x_0, x_N]$ of the domain of control under consideration, may easily be generalized to target specifically the perturbation energy at the end of the control domain, x_N. To accomplish this, we simply add to the

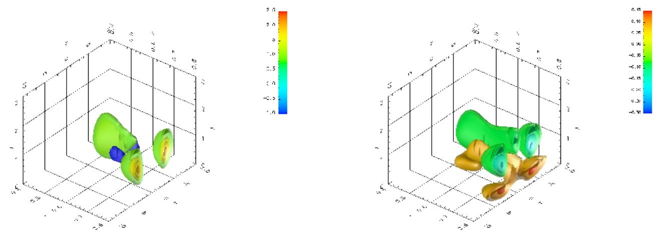

Fig. 3. Isosurfaces of $K^0(x,y,z)$ relating (left) the streamwise component of the velocity $u(x = 0.5, y, z)$, and (right) the wall–normal component of the velocity $v(x = 0.5, y, z)$ to the control input $\phi(x, z = 0)$. Control domain extension in $x = [0.5, 0.8]$. Minimization of the perturbation energy at $x = 0.8$.

cost function a penalty term on the energy of the perturbation at the end of the control domain

$$\mathcal{J} = \sum_{i=0}^{N} \Delta\left[(\mathbf{q})_i^* Q \mathbf{q}_i + \ell_\phi^2\, \phi_i^* \phi_i\right] + \sum_{i=1}^{N} \Delta\left[\ell_s^2 \,\frac{\partial \phi_i^*}{\partial x}\frac{\partial \phi_i}{\partial x}\right] + \ell_N (\mathbf{q})_N^* \Sigma_N^{11} \mathbf{q}_N, \tag{23}$$

where Σ_N^{11} is the initial condition of the Riccati equation which arises when solving the feedback control problem. We may target these outflow ("terminal") perturbations exclusively simply by setting $Q = 0$. Figure 3 represent the streamwise and wall–normal kernels in this other kind of optimization problem.

VII. NUMERICAL SIMULATIONS

By inserting these convolution feedback control kernels into a direct numerical simulation (DNS) code, we now perform simulations of the feedback controlled system, assuming full knowledge of the initial perturbation $\mathbf{q}_0^a$. Figure 1 depicts the streamwise and wall–normal kernels used in the numerical simulation. For comparison, we have also calculated the effectiveness of controls determined by applying an iterative, adjoint–based control optimization strategy, as developed by Cathalifaud and Luchini [7].
To perform the boundary-layer flow simulations, we used the spectral DNS code developed by Lundbladh et al. [14], which accurately solves the full 3D incompressible Navier-Stokes equations in the boundary layer and accounts correctly for the effects of control inputs on the wall, as thoroughly benchmarked in [14].
We have tested a worst–cade (a.k.a., "optimal") initial perturbation, that is, a perturbation whose energy is amplified maximally over the computational domain under consideration in the uncontrolled system. This kind of perturbation has been computed previously by Luchini [13], who found that such perturbations come in the form of stationary streamwise vortices, whereas the velocity field they induce is dominated by streamwise streaks. This is a typical behaviour in shear–driven flows. The control is applied over

$[x_0, x_N] = [0.5, 0.8]$, and we notice a very similar reduction of the perturbation magnitude in both the present feedback control formulation and the iterative adjoint–based control optimization.

We have also computed the energy of the perturbation $E = \int_{x_i}^{x_N} \int_{z_l}^{z_r} u^2 \; dz \quad dy$. Figure 4(a) displays the streamwise evolution of this energy. In the present feedback control formulation, the blowing/suction velocity v_w is part of the state vector $\mathbf{q}^a$. This means that, using the control law (15), the control at each streamwise station x_k depends on the velocity of blowing/suction at x_0, $v_w(x_0)$, which we impose to be zero; this leads to the control in the present formulation gently ramping up from zero at $x = x_0$. On the contrary, in the adjoint–based scheme the control $v_w(x_0)$ experiences a large jump at $x = x_0$, as shown in the figure 4(b). This explains, at least in part, the difference of effect between the two control strategies. Otherwise, the damping of the perturbation energy is found to be of the same order in both cases.

(a) (b)

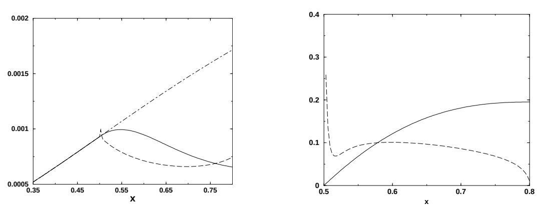

Fig. 4. (a) Evolution of the energy of perturbation E using (– · –) no control, (—) the present feedback control strategy, (– –) the iterative adjoint–based control optimization strategy of [7]. (b) Evolution of the control energy using (—) the present strategy, (– –) the adjoint–based strategy.

VIII. CONCLUSIONS

The primary challenge in the application of Riccati-based feedback control strategies to fluid-mechanical systems is the enormous state dimension which is necessary to capture such systems with an adequate degree of fidelity. The state dimension necessary to resolve such systems typically renders Riccati-based feedback control strategies numerically unfeasible.

The present paper proposes a new, Riccati-based feedback control strategy which leverages the fact that linearized boundary-layer systems develop parabolically in the streamwise coordinate. Taking advantage of this property, numerically-tractable control and estimation algorithms have been proposed which target the reduction of a globally-defined cost function with control feedback while only requiring the solution of Riccati equations related to system models which are spatially-discretized in a single coordinate direction (y).

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