A generalized framework for robust control in fluid mechanics

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1. Motivation and objectives

The application of *optimal* control theory to turbulence has proven to be quite effective when complete state information from high-resolution direct numerical simulations is available (Bewley, Moin, & Temam 1997a). In this approach, an iterative optimization algorithm based on the repeated computation of an adjoint field is used to optimize the controls for a finite-horizon nonlinear flow problem (Abergel & Temam 1990). In order to extend this infinite-dimensional optimization approach to control externally disturbed flows for which the control must be determined based on limited noisy flow measurements alone, it is necessary that the control computed be insensitive to both state disturbances and measurement noise. For this reason, *robust* control theory, a generalization of optimal control theory, is now examined as a technique by which effective control algorithms might be developed for infinite-dimensional laminar (linear) and turbulent (nonlinear) flows subjected to a wide class of external disturbances.

The numerical approach proposed to solve the robust control problem is based on computations of an $O(N)$ adjoint field, where N is the number of grid points used to resolve the continuous PDE for the flow problem. Note that $N \sim O(10^6)$ for problems of engineering interest today and may be expected to increase in the future. Computation of the adjoint field is only as difficult as the computation of the flow itself, and thus is a numerically tractable approach to the control problem whenever the computation of the flow itself is numerically tractable. In contrast, control approaches based on the solution of $O(N^2)$ Riccati equations have not been shown to be numerically tractable for discretizations with $N > O(10^3)$.

In its essence, robust control theory (Doyle *et al.* 1989, Green & Limebeer 1995) boils down to Murphy's Law (Bewley, Moin, & Temam 1997b) taken seriously:

If a worst-case system disturbance **can** *disrupt*
a controlled closed-loop system, it **will**.

When designing a robust controller, therefore, one should *plan* on a finite component of the worst-case disturbance aggravating the system, and design a controller which is suited to handle even this extreme situation. A controller which is designed to work even in the presence of a finite component of the worst-case disturbance will also be robust to a wide class of other possible disturbances which, by definition, are not as detrimental to the control objective as the worst-case disturbance. Thus, the problem of finding a robust control is intimately coupled with the problem of finding the worst-case disturbance, in the spirit of a non-cooperative game.

To summarize the robust control approach briefly, a cost functional $\mathcal{J}$ describing the control problem at hand is defined that weighs together the (distributed) control

ϕ, the (distributed) disturbance w, and the flow perturbation $u(\phi, w)$. The cost functional considered in the present work is of the form

$$\mathcal{J}(\phi, w) = \frac{1}{2}\int_0^T \int_\Omega |\mathcal{C}_1 u|^2\, dx\, dt + \frac{1}{2}\int_\Omega |\mathcal{C}_2 u(x,T)|^2\, dx + \int_0^T \int_{\partial\Omega} \mathcal{C}_3 \frac{\partial u}{\partial n}\cdot \vec{r}\, d\sigma\, dt$$
$$+ \frac{1}{2}\int_0^T \int_\Omega \left[\ell^2|\phi|^2 - \gamma^2|w|^2\right] dx\, dt.$$

This cost functional is simultaneously minimized with respect to the control ϕ and maximized with respect to the disturbance w, as illustrated in Fig. 1. The robust control problem is considered to be solved when a saddle point $(\bar{\phi}, \bar{w})$ is reached; note that such a solution, if it exists, is not necessarily unique. Four cases of particular interest are:

a. $\mathcal{C}_1 = d_1\, I$ and $\mathcal{C}_2 = \mathcal{C}_3 = 0 \Rightarrow$ regulation of the turbulent kinetic energy.
b. $\mathcal{C}_1 = d_2\, \nabla\times$ and $\mathcal{C}_2 = \mathcal{C}_3 = 0 \Rightarrow$ regulation of the square of the vorticity.
c. $\mathcal{C}_2 = d_3\, I$ and $\mathcal{C}_1 = \mathcal{C}_3 = 0 \Rightarrow$ terminal control of the turbulent kinetic energy.
d. $\mathcal{C}_3 = d_4\, \nu\, I$ and $\mathcal{C}_1 = \mathcal{C}_2 = 0 \Rightarrow$ minimization of the average skin-friction in the direction $\vec{r}$ integrated over the boundary of the domain.

All four of these cases, and many others, may be considered in the present framework; the extension to other linear/quadratic interior/boundary regulation/terminal constraints is straightforward. The dimensional constants d_i (which are the appropriate functions of the kinematic viscosity ν, a characteristic length L_0, a characteristic velocity U_0, and the volume V_0 and the surface area S_0 of the domain Ω) are included to make the cost functional dimensionally consistent.

It cannot be assumed at the outset that a solution to the min/max problem described above even exists. However, it is established in the present paper that, for a sufficiently large γ and reasonable requirements on the regularity of the problem (described later in this introduction), a solution to this min/max problem indeed does exist, with the (finite) magnitudes of the disturbance and the control governed by the scalar parameters γ and ℓ. To accomplish this, we will extend the optimal control setting of Abergel & Temam (1990) to analyze the non-cooperative differential game of the robust control setting in which a saddle point $(\bar{\phi}, \bar{w})$ is sought. The analysis will also account for the possibility of corners in the boundary Ω. Our treatment of the presence of corners in the domain avoids "smoothing" out the corners as was done in Abergel & Temam (1990) and thus further extends the optimal control analysis contained therein.

Note that, for simplicity, only the *control* problem is considered; the concomitant *estimation* problem, required to determine the control when only partial flow information is measured, is closely related to the control problem discussed here.

1.1 An intuitive introduction to robust control theory

Consider the present problem as a differential game between a fluid dynamicist seeking the "best" control ϕ which stabilizes the flow perturbation with limited

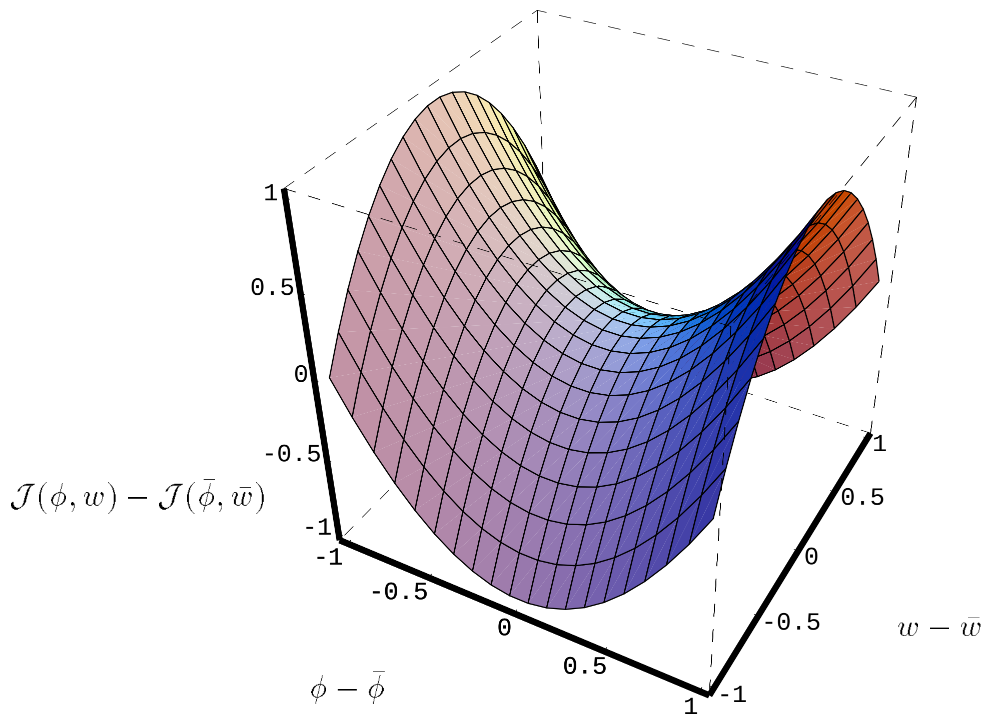


FIGURE 1. Schematic of a saddle point representing the neighborhood of a solution to a robust control problem with one scalar control variable ϕ and one scalar disturbance variable w. When the robust control problem is solved, the cost function $\mathcal{J}$ is simultaneously minimized with respect to ϕ and maximized with respect to w, and a saddle point such as $(\bar{\phi}, \bar{w})$ is reached. The present paper formulates the infinite-dimensional extension of this concept, where the cost $\mathcal{J}$ is related to a distributed control ϕ and a distributed disturbance w through the solution of the Navier-Stokes equation.

control effort and, simultaneously, nature seeking the "maximally malevolent" disturbance w which destabilizes the flow perturbation with limited disturbance magnitude (Green & Limebeer 1995). The parameter γ^2 factors into such a competition as a weighting on the magnitude of the disturbance which nature can afford to offer, in a manner analogous to the parameter ℓ^2, which is a weighting on the magnitude of the control which the fluid dynamicist can afford to offer.

The parameter ℓ^2 may be interpreted as the "price" of the control to the fluid dynamicist. The $\ell \rightarrow \infty$ limit corresponds to prohibitively "expensive" control and results in $\phi \rightarrow 0$ in the minimization with respect to ϕ for the present problem. Reduced values of ℓ increase the cost functional less upon the application of a control ϕ. A nonzero control results whenever the control ϕ can affect the flow perturbation u in such a way that the net cost functional $\mathcal{J}$ is reduced.

The parameter γ^2 may be interpreted as the "price" of the disturbance to nature. The $\gamma \rightarrow \infty$ limit results in $w \rightarrow 0$ in the maximization with respect to w, leading to the optimal control formulation of Abergel & Temam (1990) for ϕ alone. Reduced values of γ decrease the cost functional less upon the application of a disturbance w. A nonzero disturbance results whenever the disturbance w can affect the flow perturbation u in such a way that the net cost functional $\mathcal{J}$ is increased.

Solving for the control ϕ which is effective even in the presence of a disturbance w, which maximally spoils the control objective, is a way of achieving system robustness. As stated earlier, a control which works even in the presence of the malevolent disturbance w will also be robust to a wide class of other possible disturbances.

In the present systems, for $\gamma < \gamma_0$ for some critical value γ_0 (an upper bound of which is established in this paper), the non-cooperative game does not have a finite solution; essentially, the malevolent disturbance wins. The control ϕ corresponding to $\gamma = \gamma_0$ results in a stable system even when nature is on the brink of making the system unstable. However, note that the control determined with $\gamma = \gamma_0$ is not always the most suitable as it may result in a very large control magnitude and may have degraded performance in response to disturbances with structure more benign than the worst-case scenario. In the implementation, variation of ℓ and γ provide the necessary flexibility in the control design to achieve the desired trade-offs between disturbance response and control magnitude required (Bewley & Liu 1997).

1.2 Governing equations

We begin with the Navier-Stokes equation for a flow U in an open domain $\Omega \subset \mathbb{R}^3$ such that, in $\Omega \times (0, \infty)$, we have

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla) U + \nabla P = F, \\ \text{div } U = 0, \\ U = 0 \qquad \text{on } \partial\Omega, \\ U(0) = U_0 \qquad \text{at } t = 0. \end{cases} \tag{1.1}$$

We focus our attention on the case in which the forcing is applied by way of an interior volume force on the r.h.s. of the momentum equation; the case of boundary forcing (such as wall transpiration) is closely related and will be treated later. A stationary or non-stationary solution $U(x, t)$ to this equation with a corresponding forcing $F(x, t)$ will be referred to as the "target" flow for the control problem. (If no target flow is known or given, U and F are taken as zero.)

We are interested in the robust regulation of the deviation of the flow from the desired target (U, F). In §2, we consider the control of the linearized equation which models small perturbations (u, f) to the target flow (U, F) with Dirichlet boundary conditions and known initial conditions such that, in $\Omega \times (0, \infty)$, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) U + (U \cdot \nabla) u + \nabla p = f, \\ \text{div } u = 0, \\ u = 0 \qquad \text{on } \partial\Omega, \\ u(0) = u_0 \qquad \text{at } t = 0. \end{cases} \tag{1.2}$$

In §3, we consider the control of the full nonlinear equation which models large perturbations (u, f) to the target flow (U, F) such that, in $\Omega \times (0, \infty)$, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) U + (U \cdot \nabla) u + (u \cdot \nabla) u + \nabla p = f, \\ \text{div } u = 0, \\ u = 0 \qquad \text{on } \partial\Omega, \\ u(0) = u_0 \qquad \text{at } t = 0. \end{cases} \tag{1.3}$$

1.3 Mathematical setting

Let Ω be a bounded open set of $\mathbb{R}^3$ with boundary $\partial\Omega$, and let $\vec{n}$ be the unit outward normal vector to $\partial\Omega$. We denote by $H^s(\Omega)$, $s \in \mathbb{R}$ the Sobolev spaces constructed on $L^2(\Omega)$, and $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Following Temam (1984), we set $X = \left\{ u \in ((C_0^\infty(\Omega))^3;\ \text{div } u = 0 \right\}$, and denote by H (resp. V) the closure of X in $(L^2(\Omega))^3$ (resp. $(H^1(\Omega))^3$); we have

$$H = \left\{ u \in (L^2(\Omega))^3;\ \text{div } u = 0 \text{ in } \Omega, \quad u \cdot \vec{n} = 0 \text{ on } \partial\Omega \right\}$$

and

$$V = \left\{ u \in (H_0^1(\Omega))^3;\ \text{div } u = 0 \text{ in } \Omega \right\}.$$

The scalar product on H is denoted by $(u, v) = \int_\Omega u \cdot v \, dx$, that on V is denoted by $((u, v)) = \int_\Omega \nabla u \cdot \nabla v \, dx$, and the associated norms are denoted by $| \cdot |_{L^2}$ and $\| \cdot \|$ respectively. We denote by A the Stokes operator, defined as an isomorphism from V onto the dual V' of V such that, for $u \in V$, Au is defined by

$$\forall\ v \in V, \quad \langle Au, v \rangle_{V',V} = ((u, v))$$

where $\langle \cdot, \cdot \rangle_{V',V}$ is the duality bracket between V' and V. The operator A is extended to H as a linear unbounded operator with domain $D(A) = (H^2(\Omega))^3 \cap V$ when $\partial\Omega$ is a C^2 surface; the case of a domain Ω with corners is treated in §4. We also recall the Leray-Hopf projector $\mathcal{P}$, which is the orthogonal projector of the non-divergence-free space $(L^2(\Omega))^3$ onto the divergence-free space H. The Stokes operator is defined with this projector such that

$$Au = -\mathcal{P}\big(\Delta u\big), \quad \forall\ u \in D(A). \tag{1.4}$$

We shall denote by $0 < \lambda_1 \le \lambda_2 \le \ldots$ the increasing sequence of the eigenvalues of A. Define the bilinear mapping B by

$$B(u, v) = \mathcal{P}\big((u \cdot \nabla) v\big), \quad \forall\ u, v \in V. \tag{1.5}$$

Note that B is a bilinear mapping from V into V'. Define a continuous trilinear form b on V such that, with $u, v, w \in (H^1(\Omega))^3$, we have

$$\begin{aligned} b(u, v, w) &= \langle B(u, v), w \rangle_{V',V} \\ &= \int_\Omega (u \cdot \nabla) v \cdot w \, dx = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \end{aligned}$$

where Einstein's summation is assumed.

1.4 Abstract form of governing equations

The operators A and B may be used to write the Navier-Stokes equation in the "abstract form" useful for mathematical analysis. By application of the Leray projector to (1.2), noting (1.4), (1.5), and that $\mathcal{P}u = u$ and $\mathcal{P}(\nabla p) = 0$, the linearized Navier-Stokes equation, to be considered in §2, may be written in the form

$$\begin{cases} \dfrac{du}{dt} + \nu Au + B(u, U) + B(U, u) = \mathcal{P}f, \\ u \in V, \\ u(0) = u_0, \end{cases} \tag{1.6}$$

where the regularity required on f, u_0, and U are

$$\begin{cases} f \in L^2(0, T; L^2), \quad \forall\, T > 0; \\ u_0 \in V; \quad U \in \mathcal{C}([0, T], V) \cap L^2(0, T; D(A)). \end{cases} \tag{1.7}$$

Similarly, application of the Leray projector to the nonlinear form (1.3), to be considered in §3, gives

$$\begin{cases} \dfrac{du}{dt} + \nu Au + B(u, U) + B(U, u) + B(u, u) = \mathcal{P}f, \\ u \in V, \\ u(0) = u_0. \end{cases} \tag{1.8}$$

1.5 Control framework

In the control framework, the interior forcing f is decomposed into a control $\phi \in L^2(0, T, L^2)$ and a disturbance $w \in L^2(0, T, L^2)$, with $T > 0$, in the spirit of the non-cooperative game discussed in §1.1. Thus, we write f as

$$f = B_1 w + B_2 \phi, \tag{1.9}$$

where B_1 and B_2 are given bounded operators on $(L^2(\Omega))^3$. Only the divergence free part of the forcing f will affect the evolution of the velocity field u, as seen on the r.h.s. of the governing equations (1.6) and (1.8). Thus, in the remainder of this paper, we consider only the divergence free part of the forcing by writing

$$\begin{aligned} \mathcal{P}f &= \mathcal{P}(B_1 w + B_2 \phi) \\ &= \mathcal{B}_1 w + \mathcal{B}_2 \phi, \end{aligned} \tag{1.10}$$

where $\mathcal{B}_1 = \mathcal{P}B_1$ and $\mathcal{B}_2 = \mathcal{P}B_2$ are mappings from $(L^2(\Omega))^3$ to H. Note that the difference $f - \mathcal{P}f$ may be written as the gradient of a scalar and thus will only modify the pressure p in (1.2) and (1.3). As the solution to the Navier-Stokes equation in the abstract form is implicitly confined to a divergence-free manifold of $(L^2(\Omega))^3$, the pressure p may be entirely neglected in the mathematical analysis.

1.6 Important identities and inequalities

We now recall some important properties of the nonlinear operator b, which can be found, for instance, in §3 of Temam (1984). First, we have the orthogonality identity

$$b(u,v,v)=0, \quad \forall\ u,v\in V \tag{1.11}$$

as a consequence of div $u=0$, as shown by integration by parts. Moreover, the continuity of the nonlinear mapping in various functional spaces are expressed by the following classical inequalities: there exists a constant $C_0(\Omega)$ such that

$$\begin{cases} |b(u,v,w)|\le C_0\|u\|\|v\|^{1/2}|Av|_{L^2}^{1/2}|w|_{L^2}, & \forall\ u\in V,\ v\in D(A),\ w\in H,\\ |b(u,v,w)|\le C_0|u|_{L^2}^{1/4}|Au|_{L^2}^{3/4}\|v\||w|_{L^2}, & \forall\ u\in D(A),\ v\in V,\ w\in H,\ ,\\ |b(u,v,w)|\le C_0|u|_{L^2}^{1/4}\|u\|^{3/4}\|v\||w|_{L^2}^{1/4}\|w\|^{3/4}, & \forall\ u\in V,\ v\in V,\ w\in V. \end{cases}$$

where C_0 denotes here and throughout this paper a numerical constant whose value may be different in each inequality.

Note that the mapping $u\mapsto B(u)=B(u,u)$ is differentiable from V into V'; its differential is defined by

$$\begin{aligned} B'(u)v&=B(u,v)+B(v,u) \qquad \forall\ v\in V\\ &=\mathcal{P}\big((u\cdot\nabla)v+(v\cdot\nabla)u\big). \end{aligned} \tag{1.12}$$

Let $B'(u)^*$ denote the adjoint of $B'(u)$ for the duality between V and V'; the adjoint operator $B'(u)^*$ is thus defined by

$$\langle v, B'(u)w\rangle_{V,V'}=\langle B'(u)^*v,w\rangle_{V',V}\,. \tag{1.13}$$

It follows from integration by parts (Abergel & Temam 1990) that

$$\begin{aligned} \langle B'(u)^*v,w\rangle_{V',V}&=\int_\Omega\left(\frac{\partial u_i}{\partial x_j}\,v_i-\frac{\partial v_j}{\partial x_i}\,u_i\right)w_j\,dx\\ &=\int_\Omega\big((\nabla u)^T\cdot v-(\nabla v)\cdot u\big)\cdot w\,dx, \end{aligned} \tag{1.14}$$

where, again, Einstein's summation is assumed.

The use of adjoint operators to define an appropriate $O(N)$ adjoint field will be central to the development of an efficient numerical algorithm to solve the robust control problem. For the linear problem described in §2, an appropriately defined adjoint field reveals the solution $\{\bar\phi,\bar w\}$ of the robust control problem directly, as shown in §2.2. For the nonlinear problem described in §3, a solution $\{\bar\phi,\bar w\}$ of the robust control must be found by iteration, as discussed in §3.2. At each iteration k, an adjoint field is computed to determine the gradients $\mathcal{DJ}/\mathcal{D}\phi$ and $\mathcal{DJ}/\mathcal{D}w$ in the vicinity of $\{\phi^k,w^k\}$. The control ϕ^k and the disturbance w^k are then updated based on this gradient information and a new adjoint field computed until the iteration in k converges and a saddle point for the full nonlinear problem is reached. Proof of the convergence of such an algorithm is currently under development.

2. Accomplishments

As discussed in the introduction, the objective in the robust control problem is to find the best control ϕ in the presence of the disturbance w which is maximally aggravating to the control objective. The cost functional considered in the present work, in the mathematical setting described in §1.3, is given by

$$\begin{aligned}\mathcal{J}(\phi,w)=\frac{1}{2}\int_0^T \Big|\mathcal{C}_1 u\Big|^2_{L^2(\Omega)}dt+\frac{1}{2}\Big|\mathcal{C}_2 u(T)\Big|^2_{L^2(\Omega)}+\int_0^T\Big(\mathcal{C}_3\frac{\partial u}{\partial n},\vec{r}\Big)_{L^2(\partial\Omega)}dt\\ +\frac{1}{2}\int_0^T\Big[\ell^2\Big|\phi\Big|^2_{L^2(\Omega)}-\gamma^2\Big|w\Big|^2_{L^2(\Omega)}\Big]\,dt.\end{aligned} \tag{2.1}$$

where the scalar control parameters γ and ℓ are given and b is a known vector field on $\partial\Omega$. The operators $\mathcal{C}_1$ and $\mathcal{C}_2$ are unbounded operators on $(L^2(\Omega))^3$ satisfying

$$|\mathcal{C}_i u|^2_{L^2}\le\alpha|u|^2_{L^2}+\beta\|u\|^2 \quad \text{for} \quad i=1,2, \tag{2.1a}$$

with $\alpha\ge 0,\ \beta\ge 0,\ \alpha+\beta>0$, and $\mathcal{C}_3$ is a bounded operator of $(L^2(\partial\Omega))^3$, so that, by the Trace theorem (Lions & Magenes 1972), we have

$$\Big|\Big(\mathcal{C}_3\frac{\partial u}{\partial n},\vec{r}\Big)_{L^2(\partial\Omega)}\Big|\le\kappa\|u\|_{H^{3/2}}\le\kappa'\|u\|^{1/2}|Au|^{1/2}_{L^2}. \tag{2.1b}$$

where the constants κ and κ' depend upon $\vec{r}$ and Ω. In this chapter, the flow u is assumed to be related to the control ϕ and the disturbance w through the linearized Navier-Stokes equation

$$\begin{cases}\dfrac{du}{dt}+\nu Au+B(u,U)+B(U,u)=\mathcal{B}_1 w+\mathcal{B}_2\phi,\\ u\in V,\\ u(0)=u_0,\end{cases} \tag{2.2}$$

which models small deviations of the flow perturbation u from the desired target flow U. The regularity required is given by

$$\begin{cases}\phi,w\in L^2(0,T;L^2);\quad \mathcal{B}_1,\mathcal{B}_2\in\mathcal{L}(L^2,H);\\ u_0\in V;\quad U\in\mathcal{C}([0,T],V)\cap L^2(0,T;D(A)),\end{cases}$$

and the Stokes operator A, the bilinear mapping B, and other notations are described in §1.3. The robust control problem to be solved is stated precisely as:

Definition 2.1 The control $\bar\phi\in L^2(0,T,L^2)$ and disturbance $\bar w\in L^2(0,T,L^2)$, and the solution u to (2.2) associated with $\bar\phi$ and $\bar w$, are said to solve the robust control problem when a saddle point $(\bar\phi,\bar w)$ of the cost functional $\mathcal{J}$ defined in (2.1) is reached such that

$$\operatorname*{Sup}_{w\in L^2(0,T,L^2)}\mathcal{J}(\bar\phi,w)\le\mathcal{J}(\bar\phi,\bar w)\le\operatorname*{Inf}_{\phi\in L^2(0,T,L^2)}\mathcal{J}(\phi,\bar w). \tag{2.3}$$

In this chapter, we will establish both existence and uniqueness of the solution to the robust control problem stated in Definition 2.1, and will present an iterative adjoint algorithm to solve a two-point boundary value problem to find this solution.

2.1 Existence of a solution of the robust control problem

The proof of the existence of a solution $(\bar{\phi}, \bar{w})$ to the robust control problem is based on the following existence result:

Proposition 2.1. *Let $\mathcal{J}$ be a functional defined on $X \times Y$, where X and Y are non-empty, closed, convex sets. If $\mathcal{J}$ satisfies*

$$\begin{array}{ll}
\text{(a)} & \forall\ w \in Y,\ \ \phi \mapsto \mathcal{J}(\phi, w)\ \text{ is convex lower semicontinuous,} \\
\text{(b)} & \forall\ \phi \in X,\ \ w \mapsto \mathcal{J}(\phi, w)\ \text{ is concave upper semicontinuous,} \\
\text{(c)} & \exists\ w_0 \in Y\ \text{ such that } \lim\limits_{\|\phi\|_X \to +\infty} \mathcal{J}(\phi, w_0) = +\infty, \\
\text{(d)} & \exists\ \phi_0 \in X\ \text{ such that } \lim\limits_{\|w\|_Y \to +\infty} \mathcal{J}(\phi_0, w) = -\infty,
\end{array}$$

then the functional $\mathcal{J}$ has at least one saddle point $(\bar{\phi}, \bar{w})$ and

$$\mathcal{J}(\bar{\phi}, \bar{w}) = \operatorname*{Min}_{\phi \in X} \operatorname*{Sup}_{w \in Y} \mathcal{J}(\phi, w) = \operatorname*{Max}_{w \in Y} \operatorname*{Inf}_{\phi \in X} \mathcal{J}(\phi, w).$$

Proof: See §6 of Ekeland & Temam (1974).

In order to establish conditions (a) through (d) of Proposition 2.1 for the present problem, we need to analyze the evolution equation (2.2). It can be proven rigorously that, given $u_0 \in V$, $U \in \mathcal{C}([0,T], V) \cap L^2(0,T; D(A))$, and $\phi, w \in L^2(0,T; L^2)$, there exists a unique solution u of (2.2) such that

$$u \in L^2(0,T;V) \cap L^\infty(0,T,H).$$

The proof is based on the following "*a priori* estimates". Multiplying (2.2) with u, we can write

$$\begin{aligned}
\frac{d}{dt}|u|^2_{L^2} + \nu\|u\|^2 &\leq \frac{1}{\nu\lambda_1}\,|\mathcal{B}_1 w + \mathcal{B}_2\phi|^2_{L^2} + 2|b(u,U,u)| \\
&\leq \frac{1}{\nu\lambda_1}\,|\mathcal{B}_1 w + \mathcal{B}_2\phi|^2_{L^2} + C_0\|U\||u|^{1/2}_{L^2}\|u\|^{3/2},
\end{aligned}$$

Hence,

$$\frac{d}{dt}|u|^2_{L^2} + \frac{\nu}{2}\|u\|^2 \leq \frac{1}{\nu\lambda_1}\,|\mathcal{B}_1 w + \mathcal{B}_2\phi|^2_{L^2} + \frac{C_0}{\nu^6}\|U\|^4|u|^2_{L^2}.$$

Let $M_0 = \dfrac{C_0}{\nu^6} \sup\limits_{0 \leq t \leq T} \|U\|^4(t)$. Then, we have

$$|u|^2_{L^2}(t) \leq |u_0|^2_{L^2}\, e^{M_0 t} + \frac{e^{M_0 t}}{\nu\lambda_1}\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|^2_{L^2}\, ds \tag{2.4}$$

and

$$
\begin{aligned}
\frac{1}{t}\int_0^t \|u\|^2 ds &\leq \frac{2}{\nu^2\lambda_1}\frac{1}{t}\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds + \frac{2M_0}{\nu t}\int_0^t |u|_{L^2}^2 ds \\
&\leq \frac{2}{\nu^2\lambda_1 t}\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds + \frac{2}{\nu}|u_0|_{L^2}^2\, e^{M_0 T} \\
&\quad + \frac{2e^{M_0 T}}{\nu^2\lambda_1}\int_0^T |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds.
\end{aligned}
\tag{2.5}
$$

Similarly, multiplying (2.2) with Au, we can write

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2 + \nu|Au|_{L^2}^2 &\leq \frac{1}{\nu}|\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2 + 2|b(u,U,Au)| + 2|b(U,u,Au)| \\
&\leq \frac{1}{\nu}|\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2 + C_0\|U\|^{1/2}|AU|_{L^2}^{1/2}\|u\||Au|_{L^2} \\
&\qquad + C_0|U|_{L^2}^{1/4}|AU|_{L^2}^{3/4}\|u\||Au|_{L^2}.
\end{aligned}
$$

Letting

$$
M_1(T) = \frac{C_0^2}{2\nu}\sup_{0\leq t\leq T}\left(\|U\|(t) + |AU|_{L^2}(t) + |U(t)|_{L^2}^{1/2}\,|AU(t)|_{L^2}^{3/2}\right),
$$

we have

$$
\frac{d}{dt}\|u\|^2 + \frac{\nu}{2}|Au|_{L^2}^2 \leq \frac{1}{\nu}|\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2 + M_1\|u\|^2.
$$

Therefore

$$
\|u\|^2(t) \leq \|u_0\|^2\, e^{M_1 t} + \frac{1}{\nu}e^{M_1 t}\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds \tag{2.6}
$$

and

$$
\begin{aligned}
\frac{1}{t}\int_0^t |Au|_{L^2}^2 ds &\leq \frac{2}{\nu^2 t}\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds + \frac{2M_1}{t}\int_0^t \|u\|^2 ds \\
&\leq \left(\frac{2}{\nu^2 t} + \frac{4M_1}{\nu\lambda_1} + \frac{2M_1}{\nu\lambda_1}e^{M_0 T}\right)\int_0^t |\mathcal{B}_1 w + \mathcal{B}_2\phi|_{L^2}^2\, ds.
\end{aligned}
\tag{2.7}
$$

The *a priori* estimates (2.4), (2.5), (2.6), and (2.7) allow us to characterize the mapping $(\phi, w) \mapsto u(\phi, w)$. Specifically, we have:

Lemma 2.1. *For $\phi \in L^2(0,T;L^2)$, the mapping $w \mapsto u(\phi,w)$ from $L^2(0,T;L^2)$ into $L^2(0,T;V)$ is affine and continuous. Similarly, for $w \in L^2(0,T;L^2)$ the mapping $\phi \mapsto u(\phi,w)$ from $L^2(0,T;L^2)$ into $L^2(0,T;V)$ is affine and continuous. For $\phi \in L^2(0,T;L^2)$, the mapping $w \mapsto u(\phi,w)|_T$ from $L^2(0,T;L^2)$ into V is affine and continuous. Similarly, for $w \in L^2(0,T;L^2)$ the mapping $\phi \mapsto u(\phi,w)|_T$ from $L^2(0,T;L^2)$ into V is affine and continuous. Furthermore, for $u_0 \in V$ and*

$w \in L^2(0,T;L^2)$, the mapping $w \mapsto u(\phi, w)$ has a Gâteau derivative $\xi(w_1)$ in every direction $w_1 \in L^2(0,T;L^2)$, and $\xi(w_1)$ is the solution of the linear evolution equation

$$\begin{cases} \dfrac{d\xi}{dt} + \nu A\xi + B(U,\xi) + B(\xi, U) = \mathcal{B}_1 w_1, \\ \xi \in V, \\ \xi(0) = 0, \end{cases} \tag{2.8}$$

and it follows that $\xi \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$.

Proof. The fact that $w \mapsto u(\phi, w)$ and $\phi \mapsto u(\phi, w)$ are affine and continuous follows from the linearity of (2.2) and the *a priori* estimates (2.4), (2.5), (2.6), and (2.7). The existence of the Gâteau derivative as well as its characterization by (2.8) is proved in Abergel & Temam (1990), to which we refer the reader for more details.

Remark 2.1. *The solution ξ of (2.8) can be expressed as a function of w_1 in terms of the Green-Oseen's tensor $G(x,t,x',t')$ (see* Ladyzhenskaya 1969*); vaguely, we write*

$$\xi(x,t) = \int_0^T \int_\Omega G(x,t,x',t')\, w_1(x',t')\, dx'\, dt' \equiv G \cdot w_1.$$

Notationally, we will denote G by $\mathcal{D}u/\mathcal{D}w$ and $\xi(w_1)$ by $(\mathcal{D}u/\mathcal{D}w) \cdot w_1$. Note that the Green-Oseen's tensor $G = \mathcal{D}u/\mathcal{D}w$ is an infinite-dimensional extrapolation of the Jacobian of a finite-dimensional discretization of u with respect to a finite-dimensional discretization of w, as suggested by this notation. By causality, $G(x,t,x',t') = 0$ for $t' > t$.

With Lemma 2.1 established, we are ready to prove that conditions (a) through (d) of Proposition 2.1 are indeed satisfied for the present robust control problem:

Lemma 2.2. *Let $u_0 \in V$. There exists γ_0 such that, for $\gamma \geq \gamma_0$, we have*

(A) $\forall\, w \in L^2(0,T;L^2)$, $\phi \mapsto \mathcal{J}(\phi, w)$ is convex lower semicontinuous,

(B) $\forall\, \phi \in L^2(0,T;L^2)$, $w \mapsto \mathcal{J}(\phi, w)$ is concave upper semicontinuous,

(C) $\displaystyle\lim_{|\phi|_{L^2(0,T;L^2)} \to +\infty} \mathcal{J}(\phi, 0) = +\infty,$

(D) $\displaystyle\lim_{|w|_{L^2(0,T;L^2)} \to +\infty} \mathcal{J}(0, w) = -\infty.$

Proof. Condition (A): by Lemma 2.1, the map $\phi \mapsto \mathcal{J}(\phi, w)$ is lower semicontinuous. As $\phi \mapsto u(\phi, w)$ is affine, the convexity of $\phi \mapsto \mathcal{J}(\phi, w)$ follows promptly.

Condition (B): by Lemma 2.1, the map $w \mapsto \mathcal{J}(\phi, w)$ is upper semicontinuous. In order to prove concavity, note that it is sufficient to show that

$$h(\alpha) = \mathcal{J}(\phi, \alpha w_1 + w_2)$$

is concave w.r.t. α, i.e., $h''(\alpha) \leq 0$. To this end, we compute

$$h'(\alpha) = \int_0^T \Big(\mathcal{C}_1 u, \mathcal{C}_1 \frac{\mathcal{D}u}{\mathcal{D}w} \cdot w_1\Big)_{L^2(\Omega)} dt + \Big(\mathcal{C}_2 u(T), \mathcal{C}_2 \frac{\mathcal{D}u(T)}{\mathcal{D}w} \cdot w_1\Big)_{L^2(\Omega)}$$
$$+ \int_0^T \Big(\mathcal{C}_3 \frac{\partial}{\partial w} \frac{\mathcal{D}u}{\mathcal{D}w} \cdot w_1, \vec{r}\Big)_{L^2(\partial\Omega)} dt - \gamma^2 \int_0^T \Big(\alpha w_1 + w_2, w_1\Big)_{L^2(\Omega)} dt.$$

It is clear that $\xi(w_1) = (\mathcal{D}u/\mathcal{D}w)\cdot w_1$ is independent of α. Therefore,

$$h''(\alpha) = \int_0^T \left|\mathcal{C}_1 \frac{\mathcal{D}u}{\mathcal{D}w}\cdot w_1\right|_{L^2}^2 dt + \left|\mathcal{C}_2 \frac{\mathcal{D}u(T)}{\mathcal{D}w}\cdot w_1\right|_{L^2}^2 - \gamma^2 \int_0^T |w_1|_{L^2}^2 \, dt.$$

Note that $\xi(w_1)$ satisfies (2.8) by Lemma 2.1. Hence, using the *a priori* estimates (2.4), (2.5), (2.6), and (2.7), we have

$$\begin{aligned}\int_0^T \left|\mathcal{C}_1 \frac{\mathcal{D}u}{\mathcal{D}w}\cdot w_1\right|_{L^2}^2 dt \le \alpha \int_0^T |\xi|_{L^2}^2 dt + \beta \int_0^T \|\xi\|^2 \, dt \le k_1 \int_0^T |\mathcal{B}_1 w_1|_{L^2}^2 \, dt \\ \le k_1 \, |\mathcal{B}_1|_{\mathcal{L}(L^2,H)} \int_0^T |w_1|_{L^2}^2 \, dt,\end{aligned}$$

and, similarly,

$$\left|\mathcal{C}_2 \frac{\mathcal{D}u(T)}{\mathcal{D}w}\cdot w_1\right|_{L^2}^2 \le k_1 \, |\mathcal{B}_1|_{\mathcal{L}(L^2,H)} \int_0^T |w_1|_{L^2}^2 \, dt.$$

Now under the assumption that

$$\gamma^2 \ge 2k_1 \, |\mathcal{B}_1|_{\mathcal{L}(L^2,H)}\,,$$

we have $h''(\alpha) \le 0$ for $\alpha \in \mathbb{R}$. Thus the function h is concave, and the concavity of $w \mapsto \mathcal{J}(\phi, w)$ follows immediately.

Condition (C): Using (2.1b), we can write

$$\mathcal{J}(\phi, 0) \ge \frac{\ell^2}{2} |\phi|_{L^2(0,T;L^2)}^2 - \kappa' \int_0^T \|u\|^{1/2} |Au|_{L^2}^{1/2} \, dt,$$

and by the *a priori* inequalities (2.4), (2.5), (2.6), and (2.7), there exists a constant $C_0 = C_0(T, \Omega, \|u_0\|)$ such that

$$\int_0^T \|u\|^{1/2} |Au|_{L^2}^{1/2} \, dt \le C_0 |\phi|_{L^2(0,T;L^2)}.$$

Hence,

$$\mathcal{J}(\phi, 0) \ge \frac{\ell^2}{2} |\phi|_{L^2(0,T;L^2)}^2 - C_0 |\phi|_{L^2(0,T;L^2)},$$

and condition (C) follows promptly.

Condition (D): it follows from (2.4) that

$$\int_0^T |\mathcal{C}_1 u|_{L^2}^2 \, dt \le \int_0^T \left(\alpha |u|_{L^2}^2 + \beta \|u\|^2\right) dt \le k_1 \left[|\phi|_{L^2(0,T;L^2)}^2 + |w|_{L^2(0,T;L^2)}^2\right] + k_2,$$

and, similarly,

$$|\mathcal{C}_2 u(T)|^2_{L^2} \leq \left(\alpha |u(T)|^2_{L^2} + \beta \|u(T)\|^2\right) \leq k_1 \left[|\phi|^2_{L^2(0,T;L^2)} + |w|^2_{L^2(0,T;L^2)}\right] + k_2.$$

Thus, if $\gamma^2 \geq 4(k_1 + k_2)$ and $|w|_{L^2(0,T;L^2)} \geq 1$, we have

$$\begin{aligned}\mathcal{J}(0,w) &= \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2_{L^2}\, dt + \frac{1}{2}|\mathcal{C}_2 u(T)|^2_{L^2} + \int_0^T \int_{\partial\Omega} \mathcal{C}_3 \frac{\partial u}{\partial n} \cdot \vec{r}\, d\sigma\, dt - \frac{\gamma^2}{2}\int_0^T |w|^2_{L^2} ds \\ &\leq -\frac{\gamma^2}{4}|w|^2_{L^2(0,T;L^2)} + C|w|_{L^2(0,T;L^2)},\end{aligned}$$

which implies (D).

Putting the statements of this section together, we have established existence of a solution $(\bar{\phi}, \bar{w})$ to the robust control problem for a sufficiently large γ:

Theorem 2.1. *Assume that γ is sufficiently large so that*

$$\gamma^2 \geq 4(k_1 + k_2) \quad \text{and} \quad \gamma^2 \geq 2\, k_1\, |\mathcal{B}_1|_{\mathcal{L}(L^2,H)},$$

where

$$k_1 = \frac{2}{\nu^2} + \frac{T e^{M_0 T}}{\nu} \quad \text{and} \quad k_2 = \lambda_1\, |u_0|^2_{L^2}\, e^{M_0 T}.$$

Then there exists a saddle point $(\bar{\phi}, \bar{w})$ and $u(\bar{\phi}, \bar{w})$ such that

$$\mathcal{J}(\bar{\phi}, w) \leq \mathcal{J}(\bar{\phi}, \bar{w}) \leq \mathcal{J}(\phi, \bar{w}), \quad \forall\ \phi, w \text{ in } L^2(0,T;L^2).$$

Proof. The proof follows promptly from Lemmas 2.1 and 2.2 and Proposition 2.1.

2.2 Identification of the unique solution to the robust control problem

The existence of a saddle point $(\bar{\phi}, \bar{w})$ of the functional $\mathcal{J}$ implies that

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\bar{\phi}, \bar{w}) = 0 \quad \text{and} \quad \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}w}(\bar{\phi}, \bar{w}) = 0. \tag{2.9}$$

Define an adjoint state by the equation

$$\begin{cases} -\dfrac{d\lambda}{dt} + \nu A^*\lambda + B'(U)^*\lambda = \mathcal{C}_1^*\mathcal{C}_1\, u, \\ \lambda \in V_r = \left\{v \in (H^1(\Omega))^3;\ \text{div } v = 0 \text{ in } \Omega, \quad v = \mathcal{C}_3^*\vec{r} \text{ on } \partial\Omega\right\}, \\ \lambda(T) = \mathcal{C}_2^*\mathcal{C}_2\, u(T), \end{cases} \tag{2.10}$$

where A^* is defined by

$$\left(u, A^*\lambda\right)_{L^2} = \left(Au, \lambda\right)_{L^2} - \left(\mathcal{C}_3\frac{\partial u}{\partial n}, \vec{r}\right)_{L^2(\partial\Omega)} \quad \text{for } u \in D(A), \text{ and } \lambda \in V_r.$$

We have the following:

Lemma 2.3. *Let $U \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$, and let u be the solution of (2.2), $\xi_i(h)$, $i=1,2$, $h \in L^2(0,T;L^2)$ the solution of*

$$\begin{cases} \dfrac{d\xi_i}{dt} + \nu A\xi_i + B'(U)\xi_i = \mathcal{B}_i h \quad \text{for} \quad i=1,2, \\ \xi_i \in V, \\ \xi_i(0) = 0. \end{cases} \tag{2.11}$$

Then

$$\begin{aligned} \int_0^T \Big(\mathcal{B}_i^*\lambda, h\Big)_{L^2(\Omega)} dt = \int_0^T \Big(\mathcal{C}_1^*\mathcal{C}_1 u, \xi_i\Big)_{L^2(\Omega)} dt + \Big(\mathcal{C}_2^*\mathcal{C}_2 u(T), \xi_i(T)\Big)_{L^2(\Omega)} \\ + \int_0^T \Big(\mathcal{C}_3 \frac{\partial}{\partial n}\xi_i, \vec{r}\Big)_{L^2(\partial\Omega)} dt, \end{aligned} \tag{2.12}$$

where $\mathcal{B}_i^*$ is the adjoint of $\mathcal{B}_i$ for $i=1,2$.

Proof. The proof follows from integration by parts and the regularity of u, ξ_i and λ:

$$\begin{aligned} &\int_0^T \Big(\mathcal{C}_1^*\mathcal{C}_1 u, \xi_i\Big)_{L^2(\Omega)} dt + \Big(\mathcal{C}_2^*\mathcal{C}_2 u(T), \xi_i(T)\Big)_{L^2(\Omega)} + \int_0^T \Big(\mathcal{C}_3 \frac{\partial}{\partial n}\xi_i, \vec{r}\Big)_{L^2(\partial\Omega)} dt \\ &\quad = \int_0^T \Big(\Big[-\frac{d\lambda}{dt} + \nu A^*\lambda + B'(U)^*\lambda\Big], \xi_i\Big)_{L^2(\Omega)} dt \\ &\qquad + \Big(\lambda(T), \xi_i(T)\Big)_{L^2(\Omega)} + \int_0^T \Big(\mathcal{C}_3 \frac{\partial}{\partial n}\xi_i, \vec{r}\Big)_{L^2(\partial\Omega)} dt \\ &\quad = \int_0^T \Big(\lambda, \frac{d\xi_i}{dt}\Big)_{L^2(\Omega)} dt + \int_0^T \Big(\lambda, \nu A\xi_i\Big)_{L^2(\Omega)} dt + \int_0^T \Big(\lambda, B'(U)\xi_i\Big)_{L^2(\Omega)} dt \\ &\quad = \int_0^T \Big(\lambda, \mathcal{B}_i h\Big)_{L^2(\Omega)} dt = \int_0^T \Big(\mathcal{B}_i^*\lambda, h\Big)_{L^2(\Omega)} dt. \end{aligned}$$

Now we prove

Theorem 2.2. *Let $(\bar\phi, \bar w)$ be a solution of the robust control problem stated in Definition 2.1. Then*

$$\bar\phi = -\frac{1}{\ell^2}\mathcal{B}_2^*\bar\lambda \quad \text{and} \quad \bar w = \frac{1}{\gamma^2}\mathcal{B}_1^*\bar\lambda, \tag{2.13}$$

where $\bar\lambda$ is found from the solution $(\bar u, \bar\lambda)$ of the following coupled system:

$$\begin{cases} \dfrac{d\bar u}{dt} + \nu A\bar u + B'(U)\bar u = \Big(\dfrac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}_1^* - \dfrac{1}{\ell^2}\mathcal{B}_2\mathcal{B}_2^*\Big)\bar\lambda, \\ -\dfrac{d\bar\lambda}{dt} + \nu A^*\bar\lambda + B'(U)^*\bar\lambda = \mathcal{C}_1^*\mathcal{C}_1\bar u, \\ \bar u \in V, \ \bar\lambda \in V_r, \\ \bar u(0) = u_0 \ \text{ and } \bar\lambda(T) = \mathcal{C}_2^*\mathcal{C}_2\, u(T), \end{cases} \tag{2.14}$$

which admits a unique solution for $\gamma > \gamma_0(|\mathcal{B}_1|_{\mathcal{L}(L^2,H)}, |\mathcal{B}_2|_{\mathcal{L}(L^2,H)}, \ell)$.

Proof. A necessary condition for $(\bar\phi, \bar w)$ to be a saddle point of the functional $\mathcal{J}$ is

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\bar\phi,\bar w)\cdot h_1 = 0 \quad \text{and} \quad \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}w}(\bar\phi,\bar w)\cdot h_2 = 0, \quad \forall\ h_1 \in L^2(0,T;H).$$

Thus,

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}w}(\bar\phi,\bar w)\cdot h_1 = &\int_0^T \Big(\mathcal{C}_1 u, \mathcal{C}_1\frac{\mathcal{D}u}{\mathcal{D}w}\cdot h_1\Big)_{L^2(\Omega)} dt + \Big(\mathcal{C}_2 u(T), \mathcal{C}_2\frac{\mathcal{D}u(T)}{\mathcal{D}w}\cdot h_1\Big)_{L^2(\Omega)} \\ &+ \int_0^T \Big(\mathcal{C}_3\frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}w}\cdot h_1, \vec r\Big)_{L^2(\partial\Omega)} dt - \gamma^2\int_0^T \Big(\bar w, h_1\Big)_{L^2(\Omega)} dt = 0.\end{aligned}$$

and

$$\begin{aligned}\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}(\bar\phi,\bar w)\cdot h_2 = &\int_0^T \Big(\mathcal{C}_1 u, \mathcal{C}_1\frac{\mathcal{D}u}{\mathcal{D}w}\cdot h_2\Big)_{L^2(\Omega)} dt + \Big(\mathcal{C}_2 u(T), \mathcal{C}_2\frac{\mathcal{D}u(T)}{\mathcal{D}w}\cdot h_2\Big)_{L^2(\Omega)} \\ &+ \int_0^T \Big(\mathcal{C}_3\frac{\partial}{\partial n}\frac{\mathcal{D}u}{\mathcal{D}w}\cdot h_2, \vec r\Big)_{L^2(\partial\Omega)} dt + \ell^2\int_0^T \Big(\bar\phi, h_2\Big)_{L^2(\Omega)} dt = 0.\end{aligned}$$

Hence, by (2.12),

$$\int_0^T \Big(\mathcal{B}_1^*\bar\lambda - \gamma^2\bar w, h_1\Big)_{L^2(\Omega)} dt = 0, \quad \forall\ h_1 \in L^2(0,T;H)$$

and

$$\int_0^T \Big(\mathcal{B}_2^*\bar\lambda + \ell^2\bar\phi, h_2\Big)_{L^2(\Omega)} dt = 0, \quad \forall\ h_2 \in L^2(0,T;H),$$

which implies that (2.13) follows from the definition of the coupled system given in (2.14).

The uniqueness of the solution of the coupled system (2.14) is classical. For γ sufficiently large $[\gamma > \gamma_0(|\mathcal{B}_1|_{\mathcal{L}(L^2,H)}, |\mathcal{B}_2|_{\mathcal{L}(L^2,H)}, \ell)]$, we have $\big(\gamma^{-2}\,\mathcal{B}_1\mathcal{B}_1^* - \ell^{-2}\,\mathcal{B}_2\mathcal{B}_2^*\big)$ is positive definite. The proof of uniqueness then follows by multiplying the $\bar u$ equation by $\bar\lambda$ and the $\bar\lambda$ equation by $\bar u$, integrating between 0 and T, and then adding the two resulting equations.

2.3 Generalized framework

We now identify all possible sources of forcing in the two-point boundary-value problem (2.14) and thereby establish a generalized framework for which the approaches discussed herein can be applied to a wide variety of problems in fluid mechanics.

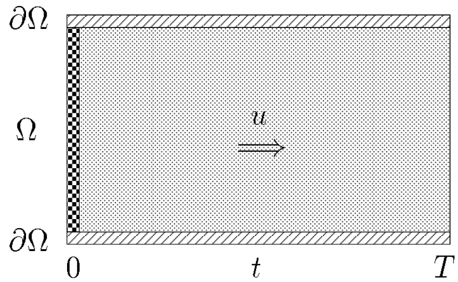
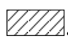


FIGURE 2. Schematic representation of the domain over which the flow field u is computed. The arrow indicates the direction in time that the p.d.e. is marched.

The space-time domain over which the flow field u is computed is illustrated in Fig. 2. The possible regions of forcing in this system are:

a. the r.h.s. of the p.d.e., indicated by , representing flow control by interior volume forcing (e.g., externally-applied electromagnetic forcing by wall-mounted magnets and electrodes);

b. the b.c.'s, indicated by , representing flow control by boundary forcing (e.g., wall transpiration);

c. the i.c.'s, indicated by , representing the optimization of the initial state in a data assimilation framework (e.g., the weather forecasting problem).

Only the first of these cases is treated in detail in the present work.

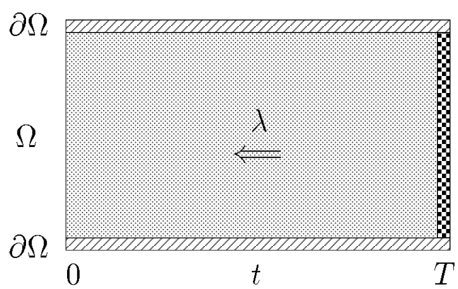
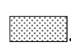
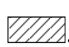


FIGURE 3. Schematic representation of the domain over which the adjoint field λ is computed. The arrow indicates the direction in time that the p.d.e. is marched.

The space-time domain over which the adjoint field λ is computed is illustrated in Fig 3. The possible regions of forcing in this system are:

a. the r.h.s. of the p.d.e., indicated by , representing regulation of an interior quantity (e.g., turbulent kinetic energy);

b. the b.c.'s, indicated by , representing regulation of a boundary quantity (e.g., wall skin-friction);

b. the i.c.'s, indicated by , representing terminal control of an interior flow quantity (e.g., turbulent kinetic energy).

All three possible locations of forcing of the adjoint problem are considered in the present framework. Note that an interesting singularity arises when considering the terminal control of a boundary quantity such as wall skin-friction. The (inhomogeneous) boundary conditions on the adjoint field for such a case are the same as in the corresponding regulation problem with a delta function applied at time $t = T$.

3. Future work

We are currently repeating the analysis of section 2 for the nonlinear problem. As mentioned in the introduction, this analysis will account for corners in the domain Ω. The analysis of existence of the solution for the nonlinear problem and the characterization of a simple gradient search routine (with fixed step size) to find this solution are both straightforward, though results are only available for a) small initial data, b) small T, or c) a 2D domain. Such a restriction is a direct consequence of the fundamental lack of a complete mathematical characterization currently available for the 3D Navier-Stokes equation, not a shortcoming of the present analysis.

In addition, we are attempting to establish rigorously the convergence of *practical* gradient search algorithms for the iterative solution of the robust control problem. To be practical, such algorithms must have *variable step size*, perhaps updating ϕ to minimize $\mathcal{J}$ in the direction $\mathcal{D}\mathcal{J}/\mathcal{D}\phi$ and/or updating w to maximize $\mathcal{J}$ in the direction $\mathcal{D}\mathcal{J}/\mathcal{D}w$ at each step of the iteration. Further, the initial guess of the solution (ϕ^0, w^0) must, in general, be considered to be "far" from the nearest solution $(\bar{\phi}, \bar{w})$ of the robust control problem. A thorough mathematical understanding of such a search algorithm is *essential* before testing these ideas numerically, as gradient searches for a saddle points even in low dimensional problems may easily get caught in limit cycles or fail altogether unless the optimization problem is thoroughly understood.

REFERENCES

ABERGEL, F., & TEMAM, R. 1990 On some control problems in fluid mechanics. *Theor. and Comp. Fluid Dynamics.* **1,** 303-325.

BEWLEY, T.R., & LIU, S. 1997 Optimal and robust control and estimation of linear paths to transition. *Submitted to J. Fluid Mech.*

BEWLEY, T.R., MOIN, P., & TEMAM, R. 1997a Optimal control of turbulence. *To be submitted to J. Fluid Mech.*

BEWLEY, T.R., MOIN, P., & TEMAM, R. 1997b Optimal and robust approaches for linear and nonlinear regulation problems in fluid mechanics. *AIAA Paper No. 97-1872.*

DOYLE, J.C., GLOVER, K., KHARGONEKAR, P.P., & FRANCIS, B.A. 1989 State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *IEEE Trans. Auto. Control.* **34,** (8), 831-847.

GREEN, M., & LIMEBEER, D.J.N. 1995 *Linear robust control.* Prentice-Hall.

LADYZHENSKAYA, O. A. 1969 *The mathematical theory of viscous incompressible flow.* Gordon & Breach.

Lions, J.L., 1969 *Quelques méthodes de résolution des problèmes nonlinéaires.* Gauthier Villars.

Temam, R. 1984 *Navier-Stokes equations*, Studies in Mathematics and its Applications 2. North-Holland.