Minimal-energy control feedback for stabilization of bluff-body wakes based on unstable open-loop eigenvalues and left eigenvectors

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An efficient technique is presented to compute minimal-energy stabilizing linear feedback control rules for linear systems. The technique presented extends easily to large-scale convection-dominated nonlinear fluid systems, linearized about unstable equilibria, as it is based solely on the least-stable eigenvalues and the corresponding left eigenvectors of the linearized open-loop system. These eigenvalues and eigenvectors, in turn, may be computed directly from a linearized simulation code via, e.g., Arnoldi or Multigrid strategies for large-scale systems. The linearized simulation code, in turn, may be computed via, e.g., the Complex Step Derivative technique from any trustworthy unsteady flow solver. Application of this procedure to a vortex-induced vibration problem resulting from the flow past a cylinder is discussed.

1 Efficient computation of minimal-energy stabilizing control feedback

It is a classical result in control theory that, if a minimal-energy stabilizing feedback control rule $\mathbf{u} = K\mathbf{x}$ is applied to the linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, the eigenvalues of the closed-loop system $A + BK$ are given by the union of the stable eigenvalues of A and the reflection of the unstable eigenvalues of A into the left-half plane across the imaginary axis. Since we know where the closed-loop eigenvalues of the system are, the requisite feedback gain matrix K in this problem may be computed by the process of *pole assignment*. Applying this process to the equation governing the dynamics of the unstable modes of the system in modal form and then transforming appropriately, this leads to a simple expression for K, as shown below.

1.1 The linear optimal control problem and its solution

Consider first the following optimization problem: for the state $\mathbf{x}$ and the control $\mathbf{u}$ related via the *state equation*

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \text{on } 0 < t < T \qquad \text{with} \qquad \mathbf{x} = \mathbf{x}_0 \quad \text{at } t = 0, \tag{1}$$

where $\mathbf{x}_0$ is initially unspecified, find the control $\mathbf{u}$ that minimizes the *cost function*

$$J = \frac{1}{2}\int_0^T \left[\mathbf{x}^H Q\mathbf{x} + \mathbf{u}^H R\mathbf{u}\right] dt \tag{2}$$

where $\{\}^H$ denotes conjugate transpose. Via standard manipulations (see, e.g., Kim & Bewley 2007), the state and relevant adjoint equations for this optimization problem may be written in the combined matrix form

$$\frac{d\mathbf{z}}{dt} = Z\mathbf{z} \quad \text{where} \quad Z = Z_{2n\times 2n} = \begin{bmatrix} A & -BR^{-1}B^H \\ -Q & -A^H \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}, \quad \text{and} \quad \begin{cases} \mathbf{x} = \mathbf{x}_0 & \text{at } t = 0, \\ \mathbf{r} = 0 & \text{at } t = T, \end{cases} \tag{3}$$

where $\mathbf{r}$ is known as the *adjoint variable*. This ODE, with both initial and terminal conditions, is a *two-point boundary value problem*. It may be solved by assuming there exists a relation between the state vector $\mathbf{x} = \mathbf{x}(t)$ and adjoint vector $\mathbf{r} = \mathbf{r}(t)$ via a matrix $X = X(t)$ such that $\mathbf{r} = X\mathbf{x}$, inserting this assumed form of the solution into the combined matrix form (3) to eliminate $\mathbf{r}$, combining rows to eliminate $d\mathbf{x}/dt$, factoring out $\mathbf{x}$ to the right, and requiring that the result holds for all $\mathbf{x}_0$, from which it follows that X obeys the *differential Riccati equation*

$$-\frac{dX}{dt} = A^H X + XA - XBR^{-1}B^H X + Q \quad \text{where} \quad X(T) = 0. \tag{4}$$

The optimal value of $\mathbf{u}$ may then be written in the form of a *feedback control rule* such that

$$\mathbf{u} = K\mathbf{x} \quad \text{where} \quad K = -R^{-1}B^H X. \tag{5}$$

Finally, if the system is linear time invariant (LTI) and we take the limit that $T \to \infty$, the matrix X in (4) may be marched to steady state. This steady state solution for X satisfies the *continuous-time algebraic Riccati equation*

$$0 = A^H X + XA - XBR^{-1}B^H X + Q, \tag{6}$$

where additionally X is constrained such that $A + BK$ is stable.

Assume now that an eigen decomposition of the composite matrix Z is available such that

$$Z = V\Lambda_c V^{-1} \quad \text{where} \quad V = \begin{bmatrix} V_{11} & * \\ V_{21} & * \end{bmatrix} = \begin{bmatrix} | & | & & | & \\ \mathbf{v}^1 & \mathbf{v}^2 & \dots & \mathbf{v}^n & * \\ | & | & & | & \end{bmatrix} \quad \text{and} \quad \mathbf{v}^i = \begin{bmatrix} \mathbf{x}^i \\ \mathbf{r}^i \end{bmatrix}, \tag{7}$$

where the eigenvalues of Z appearing in diagonal matrix Λ_c are enumerated such that the LHP eigenvalues appear first, followed by the RHP eigenvalues. Defining $\mathbf{y} = V^{-1}\mathbf{z}$, it follows from (3) that $d\mathbf{y}/dt = \Lambda_c \mathbf{y}$. The stable solutions of $\mathbf{y}$ are thus spanned by the first n columns of Λ_c (that is, they are nonzero only in the first n elements of $\mathbf{y}$). Since $\mathbf{z} = V\mathbf{y}$, it follows that the stable solutions of $\mathbf{z}$ are spanned by the first n columns of V. To achieve stability of $\mathbf{z}$ via the additional constraint $\mathbf{r} = X\mathbf{x}$ for each of these directions, denoted $\mathbf{v}^i$ and decomposed as shown above, we impose that $\mathbf{r}^i = X\mathbf{x}^i$ for $i = 1 \dots n$. Assembling these equations in matrix form, we have

$$\begin{bmatrix} | & | & & | \\ \mathbf{r}^1 & \mathbf{r}^2 & \dots & \mathbf{r}^n \\ | & | & & | \end{bmatrix} = X \begin{bmatrix} | & | & & | \\ \mathbf{x}^1 & \mathbf{x}^2 & \dots & \mathbf{x}^n \\ | & | & & | \end{bmatrix} \quad \Rightarrow \quad V_{21} = XV_{11} \quad \Rightarrow \quad X = V_{21}V_{11}^{-1}. \tag{8}$$

1.2 The minimal-energy stabilizing feedback control

Selecting any $Q > 0$ and $R = R_0/\varepsilon$ for any $R_0 > 0$ and $\varepsilon > 0$ in the above derivation, and taking the limit as $\varepsilon \to 0$, we arrive at the what is known as the *minimum-energy stabilizing feedback control*. As Z becomes block triangular in this limit, it is seen immediately that, in this limit, the eigenvalues of Z are given by the union of the eigenvalues of A and the eigenvalues of $-A^H$ for any $Q > 0$ and $R_0 > 0$. Additionally constraining this system to be stable [by the additional constraint $\mathbf{r} = X\mathbf{x}$, with X as constructed in (8)], the eigenvalues of the closed-loop system are selected precisely as the stable eigenvalues of Z; that is, the stable eigenvalues of A together with the stable eigenvalues of $-A^H$.

1.3 The pole assignment problem

Let us focus now on the eigen decomposition of Z in the above derivation:

$$\begin{bmatrix} A & -BR^{-1}B^H \\ -Q & -A^H \end{bmatrix} V_s = V_s \Lambda_{c,s} \qquad \text{with} \qquad V_s = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}, \tag{9}$$

where the n desired (stable) eigenvalues of the closed-loop system, $\lambda_{c,s}$, appear in the diagonal matrix $\Lambda_{c,s}$, and the corresponding eigenvectors of Z are given by the columns of V_s, which is partitioned as indicated. In the typical pole assignment problem, we prescribe the closed-loop eigenvalues $\lambda_{c,s}$ in advance, then modify the control input $\mathbf{u}$ [equivalently, the upper-right block of the matrix on the LHS of (9)] in order to put these eigenvalues in the desired locations. In the present pole assignment problem, however, we happen to know both the closed-loop eigenvalues $\lambda_{c,s}$ and the upper-right block of the matrix on the LHS of (9); all that remains is for us to compute the corresponding eigenvector matrix V_s. As summarized above, once these eigenvectors are calculated, the desired feedback rule is given by $\mathbf{u} = K\mathbf{x}$ with $K = -R^{-1}B^H X$, where $X = V_{21}V_{11}^{-1}$. Multiplying out (9), it follows immediately that

$$AV_{11} - BR^{-1}B^H V_{21} = V_{11}\Lambda_{c,s}, \tag{10a}$$

$$-QV_{11} - A^H V_{21} = V_{21}\Lambda_{c,s}. \tag{10b}$$

Solving (10b) for V_{11} and substituting the result into (10a) gives

$$AQ^{-1}(A^H V_{21} + V_{21}\Lambda_{c,s}) + BR^{-1}B^H V_{21} = Q^{-1}(A^H V_{21} + V_{21}\Lambda_{c,s})\Lambda_{c,s}, \tag{11a}$$

$$V_{11} = -Q^{-1}(A^H V_{21} + V_{21}\Lambda_{c,s}). \tag{11b}$$

Note that equation (11a) is linear in the unknown matrix V_{21}. Once V_{21} is obtained from this equation, calculation of V_{11} is trivial using (11b) or, equivalently, (10a).

1.4 Simplification of the linear algebra problem in modal form

It is straightforward transform the original linear system to a modal representation of its unstable dynamics. Performing the eigen decomposition $A = S\Lambda S^{-1}$ and multiplying (1) from the left by S^{-1}, it follows that

$$\dot{\boldsymbol{\chi}} = \Lambda\boldsymbol{\chi} + \bar{B}\mathbf{u} \quad \text{where} \quad \boldsymbol{\chi} = S^{-1}\mathbf{x}, \quad \bar{B} = S^{-1}B. \tag{12}$$

Note that Λ is diagonal. Denoting the inverse of the eigenvector matrix as[1] $T^H = S^{-1}$, the portion of (12) governing the unstable dynamics of the system may be written

$$\dot{\boldsymbol{\chi}}^u = \Lambda_u\boldsymbol{\chi}^u + \bar{B}_u\mathbf{u} \quad \text{where} \quad \boldsymbol{\chi}^u = T_u^H\mathbf{x}, \quad \Lambda = \begin{bmatrix}\Lambda_u & 0\\ 0 & \Lambda_s\end{bmatrix}, \quad T = \begin{bmatrix}T_u & T_s\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}\bar{B}_u\\ \bar{B}_s\end{bmatrix}, \quad \bar{B}_u = T_u^H B. \tag{13}$$

The pole assignment process in the minimal-energy stabilizing feedback control problem, as derived in §1.3, can be simplified greatly when applied to the equation for the unstable dynamics of the original system in modal form, as given in (13). Assuming A has m unstable eigenvalues, taking $A = \Lambda_u$, $B = \bar{B}_u$, $Q = I$, $R = I/\varepsilon$, and[2] $\Lambda_{c,s} = -\Lambda_u^H$ in (11a), partitioning V_{21} into its respective columns, $V_{21} = \begin{bmatrix}\mathbf{r}^1 & \mathbf{r}^2 & \dots & \mathbf{r}^m\end{bmatrix}$, and applying the above relationships, it follows after some simplifications[3] that (11a) may be written in the simple form

$$[\varepsilon\bar{B}_u\bar{B}_u^H + \mathrm{diag}(d_1^{(k)}, d_2^{(k)}, \dots, d_m^{(k)})]\mathbf{r}^k \triangleq M_k\mathbf{r}^k = 0 \quad \text{for } k = 1, \dots, m, \tag{14}$$

where

$$d_i^{(k)} = \begin{cases}(\lambda_i + \lambda_k^*)(\lambda_i^* - \lambda_k^*) & \text{for } i \neq k\\ 0 & \text{for } i = k.\end{cases} \tag{15}$$

where $\{\}^*$ denotes complex conjugate. Thus, the vectors $\mathbf{r}^k$ lie in the nullspace of M_k, and may be found by the process of Gaussian elimination, manipulating M_k to row-echelon form. In the limit $\varepsilon \to 0$, M_k approaches a diagonal matrix with a zero in the k'th diagonal element, and thus[4] $V_{21} \to I$. In order to avoid taking the difference of two quantities which are almost equal in the computation of V_{11}, we return to (10a), which, in the $\varepsilon \to 0$ limit, may be written in the form

$$\Lambda_u V_{11} + V_{11}\Lambda_u^H = \varepsilon\bar{B}_u\bar{B}_u^H \triangleq \varepsilon C. \tag{16}$$

Defining the $\{i,j\}$'th element of V_{11} as v_{ij}, the $\{i,j\}$'th element of (16) may be written $v_{ij} = \varepsilon c_{ij}/(\lambda_i + \lambda_j^*) \triangleq \varepsilon f_{ij}$. With $V_{11} = \varepsilon F$ and $V_{21} = I$, it follows that $X = F^{-1}/\varepsilon$, and thus the minimal-energy feedback control that stabilizes (13) in the limit that $\varepsilon \to 0$ is given by $\mathbf{u} = \bar{K}\boldsymbol{\chi}^u$ where $\bar{K} = -\bar{B}_u^H F^{-1}$. Writing this feedback in terms of the original state variable $\mathbf{x}$, we have $\mathbf{u} = K\mathbf{x}$ where $K = \bar{K}T_u^H$.

The solution for the minimal-energy stabilizing control feedback problem derived above is now summarized:

Theorem 1. *Consider a stabilizable system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ with no pure imaginary open-loop eigenvalues. Determine the unstable eigenvalues and corresponding left eigenvectors of A such that $T_u^H A = \Lambda_u T_u^H$ (equivalently, determine the unstable eigenvalues and corresponding right eigenvectors of A^H such that $A^H T_u = T_u\Lambda_u^H$). Define $\bar{B}_u = T_u^H B$ and $C = \bar{B}_u\bar{B}_u^H$, and compute a matrix F with elements $f_{ij} = c_{ij}/(\lambda_i + \lambda_j^*)$. The minimal-energy stabilizing feedback controller is then given by $\mathbf{u} = K\mathbf{x}$, where $K = -\bar{B}_u^H F^{-1} T_u^H$.*

[1] Note that the columns of T are referred to as the *left* or *adjoint* eigenvectors of A.

[2] We take $\Lambda_{c,s} = -\Lambda_u^H$ following the discussion in §1.2, noting that all eigenvalues in Λ_u are unstable.

[3] Note that, if Λ is diagonal, the product ΛV corresponds to scaling the i'th *row* of V by λ_i for all i, whereas the product $V\Lambda$ corresponds to scaling the i'th *column* of V by λ_i for all i.

[4] If all unstable eigenvalues of A are distinct, then $d_i^{(k)} \neq 0$ for $i \neq k$; V_{21} necessarily becomes diagonal in this case in the limit that $\varepsilon \to 0$, and its columns may be normalized such that $V_{21} \to I$. If some of the unstable eigenvalues of A are repeated, then there are other solutions as well. However, $V_{21} \to I$ is a valid solution in either case in the limit that $\varepsilon \to 0$.

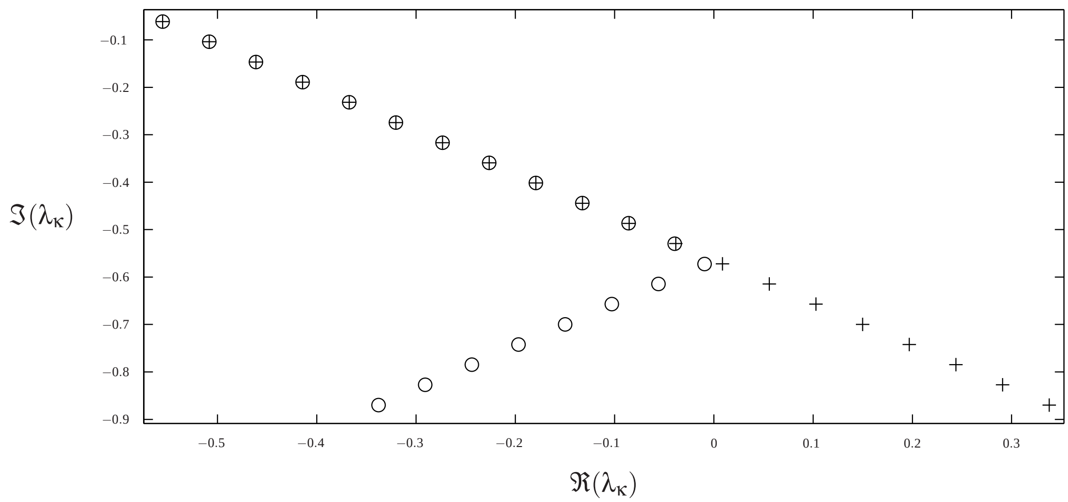

Figure 1: The first twenty eigenvalues of (+) the discretized open-loop system described in §2, and (o) the closed-loop system $A+BK$ after minimal-energy control is applied via the formulae summarized in Theorem 1.

2 Numerical results

The above algorithm was applied to the following forced convection-diffusion model of weakly nonparallel flows (see Lauga & Bewley 2003, Chomaz *et al.* 1987, 1990)

$$\frac{\partial\psi}{\partial t}+U\frac{\partial\psi}{\partial x}=\mu(x)\psi+\nu\frac{\partial^2\psi}{\partial x^2}+\delta^\sigma(x-x_f)u \qquad \Leftrightarrow \qquad \frac{\partial\psi}{\partial t}=\mathcal{L}\psi+\delta^\sigma(x-x_f)u, \tag{17}$$

where $U=6$, $\mu(x)=\mu_0-[\varepsilon(x-x_t)]^2$, $\nu=1-10\mathrm{i}$, $\varepsilon=0.01$, $x_t=0.1\mathrm{i}$, $x_f=47$ and $\delta^\sigma(x)$ is a numerical (triangular) approximation of a dirac delta representing pointwise forcing on the system. We have taken the supercriticality $(\mu_0-\mu_c)/\mu_c=3$ in the numerical simulation, where $\mu_c\triangleq\mu_a+\varepsilon\Re(\nu^{1/2})$ and $\mu_a\triangleq U^2\Re(\nu)/(4|\nu|^2)$. Results are shown in Figure 1, illustrating that the formulae provided above successfully reflect the unstable eigenvalues of A into the LHP and leave the stable eigenvalues of A unchanged. Application of this approach to a 2D cylinder wake, using our recently-developed large-scale eigenvalue solver for this class of problems, will be presented at the conference.

References

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