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On the applicability of linear feedback for nonlinear systems in fluid mechanics

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This paper examines the application of linear optimal control theory to a low-order nonlinear chaotic convection problem. Linear control feedback is found to be fully effective only when it is switched off while the state is far from the desired equilibrium point, relying on the attractor of the system to bring the state into a neighborhood of the equilibrium point before control is applied. Linear estimator feedback is found to be fully effective only when *a*) the Lyapunov exponent of the state estimation error is negative, indicating that the state estimate converges to the uncontrolled state, and *b*) the estimator is stable in the vicinity of the desired equilibrium point.

The aim in studying the present problem is to understand better some possible pitfalls of applying linear feedback to nonlinear systems in a low-dimensional framework. Such an exercise foreshadows problems likely to be encountered when applying linear feedback to infinite-dimensional nonlinear systems such as turbulence. It is important to understand these problems and the remedies available in a low-dimensional framework before moving to more complex systems such as turbulence.

I. INTRODUCTION

By major simplification of a buoyancy-driven flow problem[1] governed by the Navier-Stokes equation, a simple set of nonlinear ordinary differential equations (the Lorenz equations) which models a fluid convection problem and exhibits chaotic behavior may be determined such that

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= -x_2 - x_1 x_3 \\ \dot{x}_3 &= -b\,x_3 + x_1 x_2 - b\,r, \end{aligned} \tag{1}$$

where x_1 is proportional to the intensity of the fluid motion, x_2 is proportional to the lateral temperature fluctuations in the fluid, and x_3 is proportional to the vertical temperature fluctuations in the fluid. (In this paper, all computations are carried out for parameter values typical for a laboratory-scale implementation[2] in the chaotic regime, nominally, $\sigma = 4$, $b = 1$, and $r = 3r_H = 48$.)

Interest in this convection model has been rekindled recently by attempts to control chaotic phenomena. In the present control problem, a steady-state heating rate $\bar{u}$ is modulated by an unsteady control u' such that $r = \bar{u} + u'$. The control problem considered is to find a control u' (modulation of the cooling/heating rate at the top/bottom of the apparatus) based on limited observations of the state (specifically, noisy measurements of x_2) in order to stabilize the focus point corresponding to time-invariant clockwise motion of the fluid $\bar{\mathbf{x}}$, which is stationary but linearly unstable in the uncontrolled ($u' = 0$) Lorenz system for $\bar{u} > r_H$. This model control problem, introduced in the linear optimal context by Vincent[3] and Yuen & Bau[4], has been the topic of several recent investigations[2–8]. The present study characterizes certain problems which arise when linear feedback is used for estimation and control of this system.

State disturbances are inevitable in the present system, and come from sources such as unmodeled heat transfer and secondary flows. Noise of some level in the measurement is also inevitable, and arises from inaccuracies of the thermocouples measuring the temperature difference x_2 and from the electronics processing their signals. These disturbances are accounted for using the scaling developed in Ref. 9 by assuming that the covariance of the state disturbance has unit maximum singular value (taken here as simply $G_1 = I$) and that the rms amplitude of the noise of the (scalar) temperature measurement is α. The externally-disturbed system equation for $\dot{\mathbf{x}}$ and $\mathbf{y}$ are written in matrix form as

$$\dot{\mathbf{x}} = A\,\mathbf{x} + N(\mathbf{x}) + B_1\,\mathbf{w} + B_2\,\mathbf{u} + \mathbf{r} \tag{2a}$$

$$\mathbf{y} = C_2\mathbf{x} \qquad\qquad + D_{21}\mathbf{w}. \tag{2b}$$

with $D_{21} \triangleq (0 \;\; \alpha I)$.

II. LINEAR CONTROL FEEDBACK

In this section, we present an effective control strategy for the nonlinear system (2a) when full state information is available for determining the control. Initially, linear control theory is used to compute control feedback which linearly stabilizes the desired state. Subsequently, the resulting (linear) control feedback is applied to the full nonlinear problem.

Define the perturbation $\boldsymbol{\xi}$ of the state $\mathbf{x}$ from the desired state $\bar{\mathbf{x}}$ such that $\boldsymbol{\xi} \triangleq \mathbf{x} - \bar{\mathbf{x}}$. The stabilization of uniform clockwise motion is equivalent to the regulation of $\boldsymbol{\xi}$ to zero. Following the approach of standard linear optimal control theory, the problem under consideration is expressed as the minimization of a control objective $\mathcal{J}(\mathbf{u})$ with respect to the control $\mathbf{u}$, where

$$\mathcal{J} \triangleq E\Big[\mathbf{z}^*\mathbf{z}\Big], \quad \text{with} \;\; \mathbf{z} \triangleq C_1\boldsymbol{\xi} + D_{12}\mathbf{u}, \;\; D_{12} \triangleq \begin{pmatrix} 0 \\ \ell I \end{pmatrix}.$$

As all elements of ξ are similarly scaled, we take $C_1 = I$.

The equation governing the state perturbation $\boldsymbol{\xi}$ (in fact, for any reference point $\bar{\mathbf{x}}$) is easily derived[1] from (2a) and written in matrix form as

$$\dot{\boldsymbol{\xi}} = \bar{A}\,\boldsymbol{\xi} + N(\boldsymbol{\xi}) + B_1\,\mathbf{w} + B_2\,\mathbf{u} + \bar{\mathbf{r}}, \tag{3}$$

where $\bar{A}$ is the linearized system matrix. Note that $\bar{\mathbf{r}} = \mathbf{0}$ because $\bar{\mathbf{x}}$ is taken here to be a stationary point of the uncontrolled system (1). For sufficiently small perturbations $\boldsymbol{\xi}$, the nonlinear term $N(\boldsymbol{\xi})$ is small compared with

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the linear terms. Thus, for a state $\mathbf{x}$ in a sufficiently small neighborhood of the desired state $\bar{\mathbf{x}}$, the optimum control may be determined by analysis of just the linear terms of (3).

Linear control feedback of the form

$$\mathbf{u} = K\boldsymbol{\xi} = K(\mathbf{x} - \bar{\mathbf{x}}) \tag{4}$$

solving the minimization problem discussed above for the linearization of the system (3) governing the state perturbation $\boldsymbol{\xi}$ is given by

$$K = -\frac{1}{\ell^2} B_2^* X, \quad \text{with} \quad X = \text{Ric}\begin{pmatrix} \bar{A} & -\frac{1}{\ell^2} B_2 B_2^* \\ -C_1^* C_1 & -\bar{A}^* \end{pmatrix}$$

and Ric(·) denotes the solution of the associated Riccati problem, in accordance with standard linear optimal control theory[10].

For control feedback determined from (4) corresponding to small values of ℓ, direct application of linear feedback stabilizes both the desired state $\bar{\mathbf{x}}$ (indicated by the black trajectories of figure 1) and an undesired state $\bar{\mathbf{x}}_c'$ (indicated by the green trajectories of figure 1). An unstable manifold exists between these two stabilized points, as indicated by the contorted blue/red surfaces in figure 1. Any initial state on the blue side of this manifold will converge to the desired state, and any initial state on the red side of this manifold will converge to the undesired state.

As seen in figure 1, for increased feedback magnitude K (e.g., decreased ℓ), the undesired stabilized state $\bar{\mathbf{x}}_c'$ moves farther from the origin, and the domain of convergence of the undesired state remains large; the closed-loop system eventually becomes unbounded for sufficiently large feedback K. Some form of nonlinearity in the feedback rule is required to eliminate this undesired behavior. An effective technique tested in this study is to apply control of the form

$$\mathbf{u} = H(R - |\mathbf{x} - \bar{\mathbf{x}}|)\, K\boldsymbol{\xi}, \qquad H(\zeta) = \begin{cases} 0 & \text{for } \zeta \le 0 \\ 1 & \text{for } \zeta > 0, \end{cases}$$

such that the control is turned on only when the state $\mathbf{x}(t)$ is inside a sphere of radius R, centered at $\bar{\mathbf{x}}$, completely contained in the domain of convergence of the desired stationary point in the linearly-controlled system. The chaotic dynamics of the uncontrolled system will bring the system into this subdomain in finite time, after which control may be applied to "catch" the state at the desired equilibrium point. Similar switched approaches have been recommended by Vincent & Yu[6], Wang & Abed[7], and Vincent[3]. The key to the effectiveness of this approach is the determination a feedback control which makes the subdomain in which the linear control may be applied successfully as large as possible, so that the uncontrolled state $\mathbf{x}(t)$, moving along the attractor of the system, enters this subdomain in a short amount of time[3].

III. LINEAR ESTIMATOR FEEDBACK

When full state information is not available, one may first develop a state estimate based on the available measurements, then feed this state estimate back through a full-state controller. This chapter discusses how to determine an accurate state estimate in the present problem.

Since the state equation (2a) and the measurement equation (2b) are well known in the present problem, we will model them closely in our estimator equations such that

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + N(\hat{\mathbf{x}}) + B_2\mathbf{u} + \mathbf{r} - \hat{\mathbf{u}} \tag{5a}$$
$$\hat{\mathbf{y}} = C_2\hat{\mathbf{x}}. \tag{5b}$$

As in the previous section, the estimator feedback will be determined by application of linear theory, though this feedback is applied, in the end, to the *nonlinear* estimator given by (5).

Consider the deviations $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\eta}}$ of the state $\mathbf{x}$ and the state estimate $\hat{\mathbf{x}}$ from some (as yet undetermined) reference state $\tilde{\mathbf{x}}$ such that

$$\boldsymbol{\eta} \triangleq \mathbf{x} - \tilde{\mathbf{x}} \quad \text{and} \quad \hat{\boldsymbol{\eta}} \triangleq \hat{\mathbf{x}} - \tilde{\mathbf{x}}. \tag{6}$$

The equations governing $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\eta}}$ are easily derived from (2a) and (5a) such that

$$\dot{\boldsymbol{\eta}} = \tilde{A}\boldsymbol{\eta} + N(\boldsymbol{\eta}) + B_1\mathbf{w} + B_2\mathbf{u} + \tilde{\mathbf{r}} \tag{7a}$$
$$\dot{\hat{\boldsymbol{\eta}}} = \tilde{A}\hat{\boldsymbol{\eta}} + N(\hat{\boldsymbol{\eta}}) \qquad + B_2\mathbf{u} + \tilde{\mathbf{r}} - \hat{\mathbf{u}}, \tag{7b}$$

where $\tilde{A}$ is the linearized system matrix and $\tilde{\mathbf{r}} \triangleq A\tilde{\mathbf{x}} + N(\tilde{\mathbf{x}}) + \mathbf{r}$. Note that $\tilde{\mathbf{x}}$ need not be a stationary point, and thus $\tilde{\mathbf{r}}$ is not necessarily zero. Defining the estimation error $\mathbf{x}_e \triangleq \mathbf{x} - \hat{\mathbf{x}} = \boldsymbol{\eta} - \hat{\boldsymbol{\eta}}$ and the measurement error $\mathbf{y}_e \triangleq \mathbf{y} - \hat{\mathbf{y}}$ and subtracting (7b) from (7a) and (5b) from (2b), it is seen that $\mathbf{x}_e$ and $\mathbf{y}_e$ obey the equations

$$\dot{\mathbf{x}}_e = \tilde{A}\mathbf{x}_e + N(\boldsymbol{\eta}) - N(\hat{\boldsymbol{\eta}}) + B_1\mathbf{w} + \hat{\mathbf{u}} \tag{8a}$$
$$\mathbf{y}_e = C_2\mathbf{x}_e \qquad + D_{21}\mathbf{w}. \tag{8b}$$

The nonlinear term in this equation may be written

$$N(\boldsymbol{\eta}) - N(\hat{\boldsymbol{\eta}}) = M(\boldsymbol{\eta})\mathbf{x}_e - N(\mathbf{x}_e). \tag{9}$$

For sufficiently small $\boldsymbol{\eta}$ and $\mathbf{x}_e$, the linear terms of (8a) dominate the nonlinear term $N(\boldsymbol{\eta}) - N(\hat{\boldsymbol{\eta}})$ (see (9)). Thus, for sufficiently small estimator error $\mathbf{x}_e$ and for the state $\mathbf{x}$ in a sufficiently small neighborhood of the reference state $\tilde{\mathbf{x}}$, the estimator feedback $\hat{\mathbf{u}}$ minimizing the estimation error $\mathbf{x}_e$ may be determined by analysis of just the linear terms of (8).

Linear estimator feedback of the form

$$\hat{\mathbf{u}} = L\mathbf{y}_e = L(\mathbf{y} - \hat{\mathbf{y}}) \tag{10}$$

solving the appropriate minimization problem for the linearization of the system (8a) governing the estimation error $\mathbf{x}_e$ is given with

$$L = -\frac{1}{\alpha^2} Y C_2^*, \quad \text{with} \quad Y = \text{Ric}\begin{pmatrix} \tilde{A}^* & -\frac{1}{\alpha^2} C_2^* C_2 \\ -B_1 B_1^* & -\tilde{A} \end{pmatrix}$$

in accordance with the standard linear theory[10] for the Kalman-Bucy filter.

By applying the linear measurement feedback (10) to the undisturbed (i.e., $\mathbf{w} = \mathbf{0}$) estimation error equations (8), noting (9), the closed-loop equation for the estimation error may be written in the form

$$\dot{\mathbf{x}}_e = \left(A + LC_2 + M(\mathbf{x}(t))\right) \mathbf{x}_e - N(\mathbf{x}_e). \qquad (11)$$

Unfortunately, it does not appear possible to select time-invariant linear estimator feedback L such that the estimator error decreases uniformly as the uncontrolled state $\mathbf{x}(t)$ moves along the trajectory of the attractor, as the term $M(\mathbf{x}(t))$ is destabilizing over a portion of the attractor. However, this does not imply that the estimator will necessarily diverge; effective estimators may still be found, as will now be shown.

The convergence or divergence of the state estimator for the uncontrolled system when the estimation error $\mathbf{x}_e$ is small is now made precise. Consider an infinitesimal perturbation $\delta\mathbf{x}_e(0)$ of the state estimator such that $|\delta\mathbf{x}_e(0)| = |\mathbf{x}(0) - \hat{\mathbf{x}}(0)| \ll 1$. The perturbation $\delta\mathbf{x}_e(t)$ evolves according to the linearization of (11), which is given by

$$\dot{\delta\mathbf{x}}_e = \left(A + LC_2 + M(\mathbf{x}(t))\right) \delta\mathbf{x}_e.$$

The Lyapunov exponent κ_∞ is defined as

$$\kappa_\infty = \lim_{T\to\infty} \frac{1}{T} \log \frac{\|\delta\mathbf{x}_e(T)\|}{\|\delta\mathbf{x}_e(0)\|}$$

for almost all initial states $\mathbf{x}(0)$ and initial infinitesimal estimator perturbations $\delta\mathbf{x}_e(0)$, in a manner analogous to the Lyapunov exponent of the uncontrolled system[11]. The Lyapunov exponent of the state estimation error, κ_∞, thus measures the exponential rate of convergence ($\kappa_\infty < 0$) or divergence ($\kappa_\infty > 0$) of the state estimator when averaged over long time intervals ($T \to \infty$). The local Lyapunov exponent $\kappa_\epsilon(\mathbf{x}(t))$ is defined as

$$\kappa_\epsilon(\mathbf{x}(t)) = \lim_{T\to 0} \frac{1}{T} \log \frac{\|\delta\mathbf{x}_e(t+T)\|}{\|\delta\mathbf{x}_e(t)\|}$$

for almost all initial states $\mathbf{x}(0)$ and initial infinitesimal estimator perturbations $\delta\mathbf{x}_e(0)$ and for t sufficiently large, in a manner analogous to the local Lyapunov exponent $\lambda_\epsilon(\mathbf{x}(t))$ of the uncontrolled system[12]. The local Lyapunov exponent of the state estimation error, $\kappa_\epsilon(\mathbf{x}(t))$, thus measures the local exponential rate of convergence or divergence of state and the state estimate when the estimation error is small. The Lyapunov exponent κ_∞ is the long-time average along the system trajectory $\mathbf{x}(t)$ of the local Lyapunov exponent $\kappa_\epsilon(\mathbf{x}(t))$.

It is demonstrated in simulations (figure 3) that, for α sufficiently small that $\kappa_\infty < 0$, the estimator feedback stabilizes the estimation error $\mathbf{x}_e$ to zero even for initial conditions of the estimation error $\mathbf{x}_e(0)$ which are not small. As opposed to the control problem, no undesired stabilized states other than $\mathbf{x}_e = 0$ were detected in the closed-loop nonlinear system for the estimation error.

It was found (compare figures 2b and 2c) that choosing a (time-invariant) reference state $\tilde{\mathbf{x}}$ at the origin, which is the approximate "center of mass" of the orbits of the uncontrolled system, gave the best estimator performance for the range of initial conditions tested.

It was also found (compare figures 2b and 2d) that the nonlinear term $N(\hat{\mathbf{x}})$ in the estimator (5a) is essential for good estimator performance. Without it, the equation for a small perturbation $\delta\mathbf{x}_e(t)$ of the estimator (when we take $\tilde{\mathbf{x}} = \mathbf{0}$) takes the form

$$\dot{\delta\mathbf{x}}_e = \left(A + LC_2\right) \delta\mathbf{x}_e + N\left(\mathbf{x}(t)\right),$$

where the contribution of the nonlinear term $N(\mathbf{x}(t))$ is not small as the state $\mathbf{x}(t)$ moves on the attractor.

IV. FURTHER DISCUSSION

For further discussion of the present results and related questions in the turbulence problem, the reader is referred to the complete version of this paper scheduled to appear in the May 1999 issue of *Physics of Fluids.*

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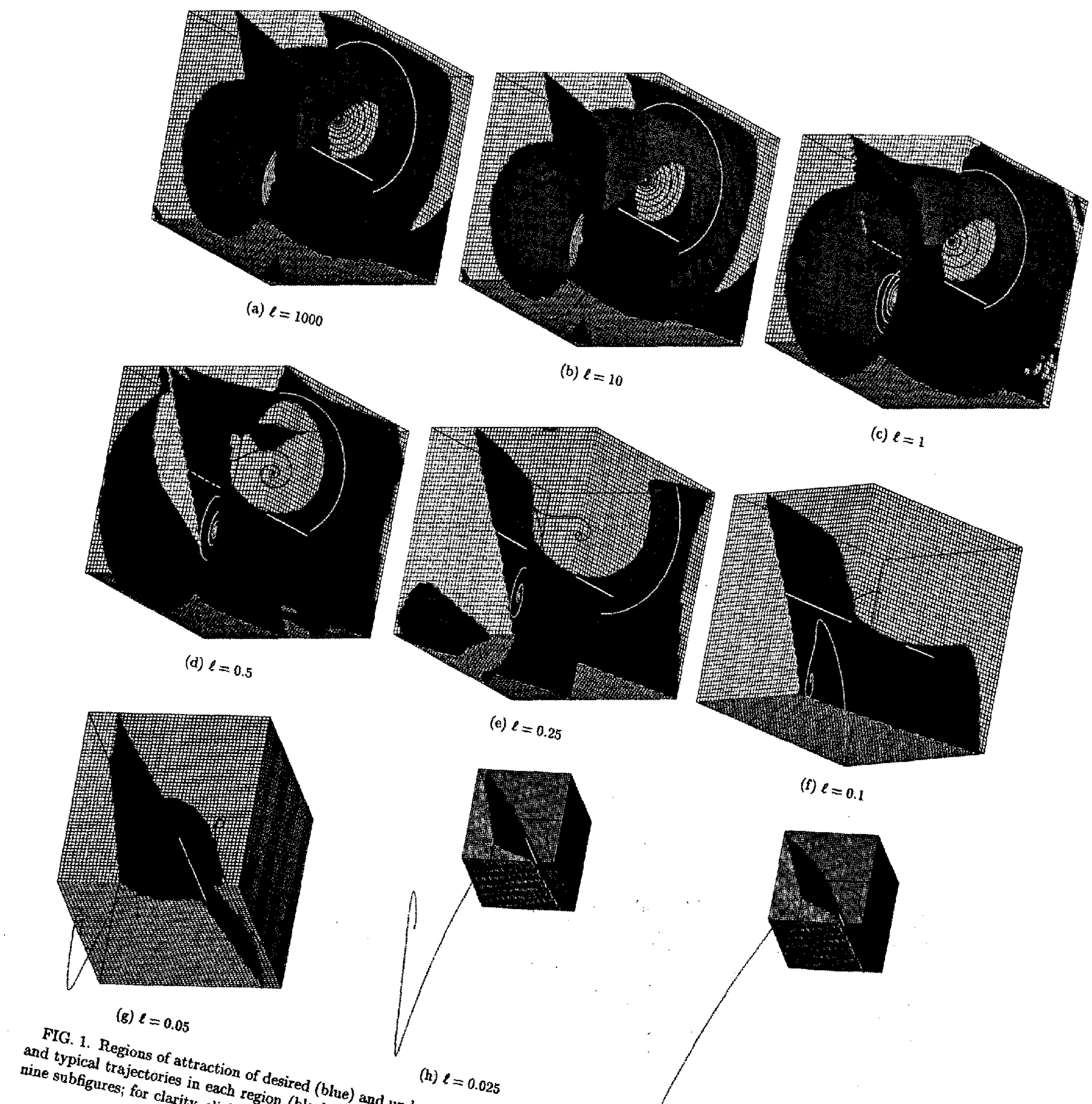


(a) $\ell = 1000$

(b) $\ell = 10$

(c) $\ell = 1$

(d) $\ell = 0.5$

(e) $\ell = 0.25$

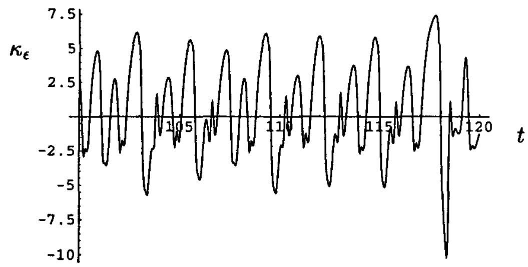
(f) $\ell = 0.1$

(g) $\ell = 0.05$

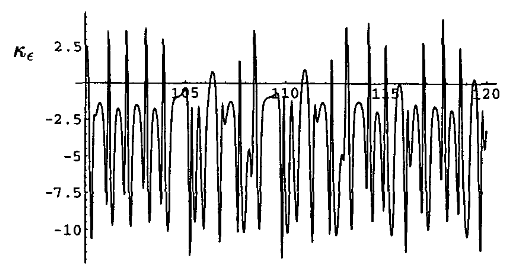
(h) $\ell = 0.025$

(i) $\ell = 0.01$ (unbounded)

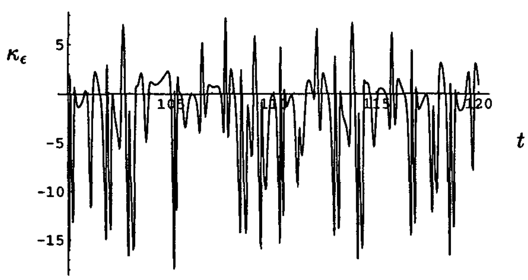
FIG. 1. Regions of attraction of desired (blue) and undesired (red) stationary points in linearly-controlled convection system and typical trajectories in each region (black and green respectively). The cubical domain illustrated is $\Omega = (-25, 25)^3$ in all nine subfigures; for clarity, slightly different viewpoints are used in each subfigure.



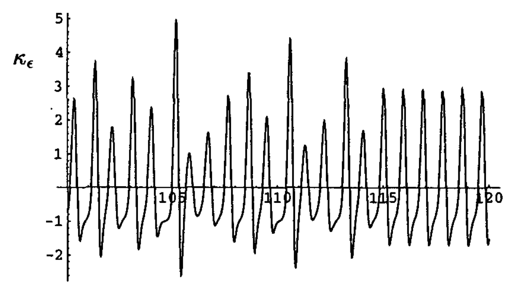
(a) Estimator designed with $\alpha = 1.0$, $\tilde{\mathbf{x}} = \mathbf{0}$. The value of the Lyapunov exponent κ_∞, which is the average value of the local Lyapunov exponent κ_ϵ plotted, is $\kappa_\infty = 0.45 > 0$. This indicates that the state estimator is unstable ($\kappa_\epsilon > 0$) more than it is stable ($\kappa_\epsilon < 0$), and thus the state estimate will not converge to the uncontrolled state.



(b) Estimator designed with $\alpha = 0.1$, $\tilde{\mathbf{x}} = \mathbf{0}$. The value of the Lyapunov exponent is $\kappa_\infty = -3.95 < 0$. This indicates that the state estimator is stable more than it is unstable, and thus the state estimate will converge to the uncontrolled state when $\mathbf{x}_e$ is small. Note that estimator convergence is attained even though the estimator error does not decrease uniformly over the entire path of the attractor.

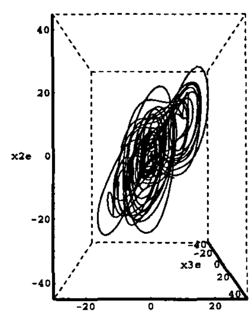


(c) Estimator designed with $\alpha = 0.1$, $\tilde{\mathbf{x}} = \bar{\mathbf{x}}$. Lyapunov exponent $\kappa_\infty = -2.33 < 0$. It is found that linear estimator feedback designed with $\tilde{\mathbf{x}} = \mathbf{0}$ has better convergence properties (cf. figure 2b).

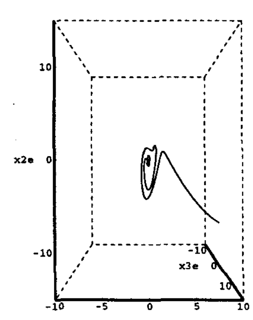


(d) Estimator designed with $\alpha = 0.1$, $\tilde{\mathbf{x}} = \mathbf{0}$, and the nonlinear term dropped from the estimator equation (5). Lyapunov exponent $\kappa_\infty = 0.01$. The nonlinear term in the estimator is essential for good performance (cf. figure 2b).

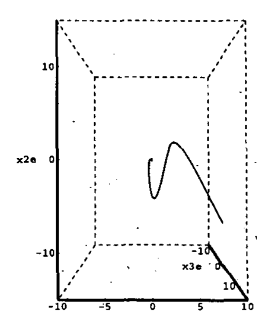
FIG. 2. Local Lyapunov exponent $\kappa_\epsilon(t)$ describing the local growth or attenuation of small perturbations of the estimation error $\mathbf{x}_e(t)$ in the closed-loop system for the state estimator as the state $\mathbf{x}(t)$ moves along the attractor.



(a) $\alpha = 1.0$, $\kappa_\infty = 0.45$.



(b) $\alpha = 0.25$, $\kappa_\infty = -1.04$.



(c) $\alpha = 0.1$, $\kappa_\infty = -3.95$.

FIG. 3. Trajectory of the estimation error $\mathbf{x}_e(t)$ for estimators determined with $\tilde{\mathbf{x}} = 0$ and three different values of α when applied to the uncontrolled, undisturbed convection system. The initial conditions on the state, $\mathbf{x}(0) = (5\ 1\ 0)^*$, and the state estimate, $\hat{\mathbf{x}}(0) = (-5\ 10\ 0)^*$, are separated significantly in these simulations. Even so, for estimators with $\kappa_\infty < 0$, the estimator feedback $\hat{\mathbf{u}}$ rapidly brings the state estimate $\hat{\mathbf{x}}$ in close proximity to the state $\mathbf{x}$ based on measurements of x_2 only. Such behavior is seen with all initial conditions tested. The approach of the estimated state to the actual state is more rapid for estimators with more negative values of κ_∞. After the state and the estimate are brought into proximity, nonlinear estimators with $\kappa_\infty < 0$ accurately track the chaotic trajectory of the state with little further estimator feedback required.