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SAFE AND RELIABLE COVERAGE CONTROL

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Abstract. In this paper we consider a problem of designing control laws for multiple mobile agents trying to accomplish three objectives. One of the objectives is to sense a given compact domain while satisfying the other objective which is to avoid collisions between the agents themselves as well as with the obstacles. To keep the communication links between the agents reliable, the agents need to stay relatively close during the sensing operation which is the third and final objective. The design of control laws is based on carefully constructed objective functions and on an assumption that the agents' dynamic models are nonlinear yet affine in control laws. As an illustration of some performance characteristics of the proposed control laws, a numerical example is provided.

1. **Introduction.** When controlling and coordinating multiple agents one of the main issues is to provide a guarantee that there will be no collisions between the agents as well as with the obstacles. Not only that the problem of collision avoidance is difficult on its own (one of the reasons being its relation to differential games [17, 2, 29]) yet the complexity significantly increases if additional objectives are considered, especially in the case when the agents' dynamical characteristics are represented by nonlinear models. Due to its relation to the Liapunov stability approach known for its applicability to control of nonlinear systems, a concept of *avoidance control* introduced by Leitmann and Skowronski [22] and later further developed by Leitmann and his collaborators in [23, 24, 25, 7, 8], provided much needed theoretical framework to tackle issues of controlling multiple nonlinear systems with collision avoidance as one of the objectives. The avoidance control framework was initially established as a Liapunov based approach to pursuit-evasion

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dynamic games yet its applicability goes well beyond its original formulation. So far the method has been successfully applied to multi-vehicle systems involving differential drive robots [26], quadrotors [44], miniature helicopters [31] as well as in modelling human behavior [3]. A more recent and comprehensive survey on both theoretical and application results of avoidance control is provided in [38].

General multi-objective problems are also known to be extremely difficult to solve. This can be understood by recalling that even in the simpler static cases (meaning no differential equations are involved) of mutiobjective optimization formulations the most common approach is to avoid vector optimizations and introduce appropriate scalarizations of the problems [28]. We follow this approach in our paper by using approximations of minimum and maximum functions used first in differential games [37, 39] and later used to construct Liapunov-like goal functions for multi-objective problems in controls [40, 41] where each objective is represented by a corresponding objective function. Collision avoidance objective is represented using modified avoidance functions [36] which are active only in bounded domains around the agents. Coverage control objective is described using an area integral [15, 16] and the agents' sensing domains are assumed to be compact circular regions. We provide additional analysis on how to differentiate area integrals dependent on time in the case when the individual agents' sensing areas overlap. This is an extension of the result valid in the case when the agents' sensing domains do not overlap [42]. Since the coverage control problem is a singular control problem [45], we construct an area integral which differs from the ones used in [15, 16, 42] because we do not assume *a priori* that the agents are forbidden to leave the search area. At this point let us recall that the origins of coverage control date back to a problem of a search of an unknown (static or mobile) object studied in the differential games literature [4, 27, 30]. Finally we formulate proximity objective functions using simple penalty function forms since the proximity objective is needed for reliability of the communication links and not treated as a main objective which was the case in, for example, [46]. In order to illustrate the proposed design we consider a scenario with three agents represented by nonlinear and nonholonomic models which are affine in control. The agents' goal is to sense a given domain which is a square (yet the design procedure does not depend on the shape of the area) while avoiding collisions and maintaining reliable communication links.

2. **Avoidance and proximity control for multiple agents.**

In this section we show how to design avoidance and proximity control laws based on appropriately constructed objective functions. We start by assuming that the agents' dynamic models are nonlinear yet affine in control so they can be written as

$$\dot{x}_i = f_i(x_i, u_i) = g_i(x_i)u_i + h_i(x_i),\ x_i(0) = x_{io},\ \forall t \in [0, +\infty),\ \forall i \in \mathbf{N} \tag{1}$$

where N denotes a number of agents, $\mathbf{N} = \{1, \ldots, N\}$, $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the control input/law, and x_{io} is a given initial condition for an i-th agent. It is relevant to note that these nonlinear affine in control models include differential drives and car-like models [34]. The n_i-dimensional vector functions $f_i(\cdot, \cdot)$, $i \in \mathbf{N}$, are assumed to be continuously differentiable with respect to both arguments. The agents' control inputs are assumed to belong to the set of admissible feedback strategies with respect to the overall state $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^n$, that is, $u_i \in \mathcal{U}_i = \{\phi_i(\cdot) : \mathbb{R}^n \to \mathbb{R}^{m_i}\}$ for all $i \in \mathbf{N}$. An admissible set of feedback strategies consists of those strategies which would guarantee both existence and uniqueness of the closed-loop state trajectories of differential equations in (1) (for more details we refer to, for example, [6, 9]).

2.1. **Avoidance objective functions.** The avoidance control objectives include avoiding collisions with static obstacles as well as collisions between the agents. A modified avoidance function to be used to avoid collisions between agents i and j has the following form [36]:

$$v_{ij}^a(x) = \left(\min\left\{0, \frac{(x_i^p - x_j^p)^T P_{ij}(x_i^p - x_j^p) - R_{ij}^2}{(x_i^p - x_j^p)^T P_{ij}(x_i^p - x_j^p) - r_{ij}^2}\right\}\right)^2 \tag{2}$$

where $R_{ij} > r_{ij}$ are positive scalars and P_{ij} is a positive definite matrix for any pair (i, j). The superscript p denotes position state variables which form a subset of the corresponding agent's state variables. Particular values for r_{ij}, R_{ij} and P_{ij} are chosen to satisfy the safety requirements which are related to particular scenarios and dynamical characteristics of the agents. In a similar way, a modified avoidance function to be used to design a control input for an agent i to avoid an obstacle denoted by a positive integer l takes the following form:

$$v_{il}^a(x) = \left(\min\left\{0, \frac{(x_i^p - x_l^o)^T P_{il}(x_i^p - x_l^o) - R_{il}^2}{(x_i^p - x_l^o)^T P_{il}(x_i^p - x_l^o) - r_{il}^2}\right\}\right)^2, \tag{3}$$

where $R_{il} > r_{il} > 0$ are positive scalars and P_{il} is a positive definite matrix corresponding to the pair (i, l). These parameters are chosen so that the set $\{z : (z - x_j^o)^T P_{ij}(z - x_j^o) \leq r_{ij}^2\}$ either exactly defines or over-bounds obstacle l, and R_{il} is chosen based on the sensing capabilities of an agent i.

The gradients of the avoidance functions v_{ij}^a are given by [36]

$$\frac{\partial v_{ij}^a}{\partial x_i^p} = \begin{cases} 0, & d_{ij} \geq R_{ij} \\ 4\frac{(R_{ij}^2 - r_{ij}^2)((x_i^p - \hat{x}_j)^T P_{ij}(x_i^p - \hat{x}_j) - R_{ij}^2)}{((x_i^p - \hat{x}_j)^T P_{ij}(x_i^p - \hat{x}_j) - r_{ij}^2)^3}(x_i^p - \hat{x}_j)^T P_{ij}, & r_{ij} < d_{ij} < R_{ij} \\ \text{not defined}, & d_{ij} = r_{ij} \\ 0, & d_{ij} < r_{ij} \end{cases} \tag{4}$$

where $\hat{x}_j$ denotes x_j^p if j corresponds to an agent or x_j^o if j corresponds to an obstacle. In addition, $\|y\|_{P_{ij}} = \sqrt{y^T P_{ij} y}$ where y is a vector and P_{ij} is a positive definite matrix of appropriate dimension. Finally, for short notation we set $d_{ij} = \|x_i^p - \hat{x}_j\|_{P_{ij}}$.

2.2. **Proximity objective functions.** During the operation we assume that the agents exchange information via wireless communication links. In order to keep these links reliable the agents need to stay close. This requirement is formulated in terms of distances between the agents being smaller than particular distances which guarantee reliable communication links. This means that any two agents i and j are accomplishing other objectives (such as coverage) while trying to keep mutual distance to be less or equal to a distance $\hat{R}_{ij}$ which allows them to communicate without interruptions. To formulate a proximity objective for agents i and j to be distance wise closer than $\hat{R}_{ij}$, we use the following function (similar to the ones used in [16]):

$$v_{ij}^p(x_i^p, x_j^p) = \max\{0, \|x_i^p - x_j^p\|^2 - \hat{R}_{ij}^2\}^2. \tag{5}$$

Then the gradient of the proximity function with respect to the i-th agent's position state variables can be computed as

$$\frac{\partial v_{ij}^p}{\partial x_i^p} = 4 \max\{0, \|x_i^p - x_j^p\|^2 - \hat{R}_{ij}^2\}(x_i^p - x_j^p)^T. \tag{6}$$

2.3. **Designing avoidance and proximity control laws.** In order to accomplish multiple objectives using a Liapunov-like analysis we adopt an approach that is based on differentiable approximations of minimum and maximum functions [39, 40, 41]. Since this paper deals with control applications (not differential games as in [39, 40, 41]) we only need an approximation of the maximum of the form

$$\overline{\rho}(\delta, a) = \sqrt[\delta]{\sum_{i=1}^{N} a_i^\delta}, \tag{7}$$

where $\delta \in \mathbb{R}_+ = (0, +\infty)$, $a = [a_1, \dots, a_N]^T \in \mathbb{R}_+^N$, and N is a positive integer. We also recall the following functions of either δ or a [40, 41]:

$$\begin{aligned} \overline{\rho}_a(\delta) &= \overline{\rho}(\delta, a) \text{ for any given } a \in \mathbb{R}_+^N, \\ \overline{\rho}_\delta(a) &= \overline{\rho}(\delta, a) \text{ for any given } \delta \in \mathbb{R}_+, \end{aligned} \tag{8}$$

where $\overline{\rho}_p(a)$ is known as a p-norm of a denoted as $\|a\|_p$ when $p = \delta \in [1, +\infty]$. In what follows, we will denote Euclidean norm without the subscript, that is, $\|\cdot\| \equiv \|\cdot\|_2$. Let $a_M = \max_{i \in \mathbf{N}}\{a_i\}$ and define M as a variable taking the integer value of the index of a maximum a_j, that is, $M = j$. Maximum approximation functions given in equation (8), when $N \geq 2$, satisfy that $a_M < \overline{\rho}_a(\delta_2) < \overline{\rho}_a(\delta_1)$, $\forall(\delta_1, \delta_2)(0 < \delta_1 < \delta_2 < +\infty)$, and also that $\lim_{\delta \to +\infty} \overline{\rho}_a(\delta) = a_M$ [40, 41].

The advantages of using approximation functions in (8) are that they can be used as an appropriate scalarization for multiobjective problems [28] as well as incorporated using a Liapunov-like analysis to establish sufficient conditions for achieving multiple objectives. Furthermore it is assumed that each objective is represented by a scalar nonnegative function $v_{ij}(\cdot) : [t_0, +\infty) \times \mathbb{R}^n \to [0, +\infty)$, where the subscript j denotes the j-th objective, the subscript i denotes the i-th agent, and t_0 denotes initial time. Objective functions are functions of time and agents' state variables, in general. Let us assume that the i-th agent's goal is to accomplish N_i objectives which are formulated as $v_{ij} \leq \epsilon_{ij}$ for some nonnegative numbers ϵ_{ij} which are chosen appropriately. For example if a proximity function in (6) satisfies $v_{ij} \leq 0$ then the distance between agents i and j is less or equal to $\hat{R}_{ij}$. Then, a sufficient condition for agent i to accomplish all objectives can be formulated as follows [40, 41]:

$$\overline{\rho}_\delta(v_{i1}, \dots, v_{iN_i}) \leq \min\{\epsilon_{i1}, \dots, \epsilon_{iN_i}\}. \tag{9}$$

In the case in which all ϵ_{ij} are positive, condition (9) can be relaxed to [40, 41]

$$\overline{\rho}_\delta(\gamma_{i1} v_{i1}, \dots, \gamma_{iN_i} v_{iN_i}) \leq 1, \text{ where } \gamma_{ij} = 1/\epsilon_{ij}, j \in \{1, \dots, N_i\}. \tag{10}$$

Notice that inequality (10) implies $v_{ij} \leq 1/\gamma_{ij} = \epsilon_{ij}$ which means that all of the agent i's objectives are satisfied.

Now let us assume that the overall goal (which is to accomplish all of its objectives) for the i-th agent is mathematically formulated as $v_i(t, x) \leq \epsilon$ where

$$v_i(\cdot) = \overline{\rho}_\delta(\gamma_{i1} v_{i1}(\cdot), \dots, \gamma_{iN_i} v_{iN_i}(\cdot)). \tag{11}$$

These objectives are formulated in terms of avoidance functions in (2) and (3) as well as proximity objective functions in (5). One way to satisfy all objectives is to decrease values of function $v_i(\cdot)$ along the trajectories of (1) in time until it satisfies the goal condition at some time $t = T$ when $v_i(T, x(T)) \leq \epsilon$ because $\overline{\rho}_\delta(\cdot)$ is always greater or equal to the maximum value of the objective functions. This approach is very appropriate in the case of multiple agent dynamic systems where the dynamics of the agents are affine in control such as in equation (1). In that case control inputs can be designed using the agents' dynamics and the gradients of their goal functions as

$$\hat{u}_i(x) = -\hat{k}_i g_i^T(x_i) \frac{\partial v_i(x)}{\partial x_i}^T \tag{12}$$

where $\hat{k}_i$ denotes a positive scalar gain.

It was shown in [37, 39] that when the agents' models are nonlinear and the goal is to accomplish multiple objectives then the goal function may not be monotonically decreasing (or nonincreasing) in time (as in the standard Liapunov functions approach) and one way to deal with the issue is to use differential inequalities to establish a sufficient condition for accomplishing the goal. To illustrate this let us assume that $v(t_0, x(t_0)) > R$ and that a goal is to achieve $v(T, x(T)) < R$, $R > 0$, as well as determine an instant of time $T > t_0$. Then let us bound the time derivative of the goal function using a function $G(\cdot,\cdot,\cdot) : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, that is,

$$\frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t, x, \hat{u}(t,x)) \leq G(t, x, v(t,x)) \tag{13}$$

for any $(t, x) \in [t_0, +\infty) \times \mathbb{R}^n$. The choice of $G(\cdot,\cdot,\cdot)$ is very much problem dependent yet it is often chosen to be equal to the time derivative of the goal function. Furthermore if the maximal solution $\bar{z}(t)$ [19, 10] of the comparison differential equation:

$$\dot{z}(t) = G(t, x(t), z(t)),\ z(t_0) = z_0 \tag{14}$$

with initial condition $z_0 \geq v(t_0, x_0)$, satisfies $\bar{z}(T) < R$ for some T, then $v(T, x(T)) \leq \bar{z}(T) < R$ and thus the goal is achieved at $t = T$.

3. **Coverage control.**

In this section we concentrate on the design of agents' coverage control laws when their dynamics are affine in control. The coverage control problem is formulated in terms of an area integral which depends on time as a parameter and the value of this integral shows how well the area is covered. The coverage itself is done with a group of agents with limited and nonuniform sensing capabilities. If one uses a Liapunov approach to establish a sufficient condition that the area is properly covered then one of the issues which arises is the differentiation of an area integral depending on time as a parameter (also known as Leibniz's integral rule). This was not treated as an issue in [15, 16] and an initial result involving one sensor or multiple sensors with non-overlapping sensing regions was provided in [42]. In this paper we provide a formula for the differentiation of an area integral which is computed over the domain in the case when agents' sensing regions do overlap. We also provide a design of the coverage control laws for agents with nonlinear models affine in control based on a modified area integral (comparing to the one used in [15, 16, 42]). This modified area integral enables the agents to search an area without imposing an additional constraint which is not allowing the agents to leave the area.

3.1. **Differentiation of double integrals depending on a parameter.** In this subsection we recall some known results and provide some new derivations related to the differentiation of area integrals related to our coverage control formulation. Let us first assume that $f(t,\tilde{z}_1,\tilde{z}_2)$ and $\frac{\partial f(t,\tilde{z}_1,\tilde{z}_2)}{\partial t}$ are continuous functions in the domain $\{(t,\tilde{z}_1,\tilde{z}_2) : (t,\tilde{z}_1,\tilde{z}_2) \in [t_1,t_2] \times D(t)\}$. Constants t_1 and t_2 are given. Now we will recall some results for the case when $D(t)$ is a compact region in $\mathbb{R}^2$ bounded by a smooth Jordan curve $C(t)$ which were reported in [42]. We start by considering the time derivative of the following area integral:

$$J(t) = \iint\limits_{D(t)} f(t,\tilde{z}_1,\tilde{z}_2) d\tilde{z}_1 d\tilde{z}_2 \tag{15}$$

which can be computed as [11, 21]

$$\frac{dJ(t)}{dt} = \frac{d}{dt}\left(\iint\limits_{D(t)} f(t,\tilde{z}_1,\tilde{z}_2) d\tilde{z}_1 d\tilde{z}_2\right) = \iint\limits_{D(t)} \frac{\partial f(t,\tilde{z}_1,\tilde{z}_2)}{\partial t} d\tilde{z}_1 d\tilde{z}_2 + I(t) \tag{16}$$

where

$$I(t) = \oint\limits_{C(t)} f(t,\tilde{z}_1,\tilde{z}_2)\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \tag{17}$$

and the integration is done in the counterclockwise (that is, positive) direction. It fits our scenario to assume that for each t the region $D(t)$ is a circle, that is,

$$\begin{aligned} D(t) &= \{(\tilde{z}_1,\tilde{z}_2) : (\tilde{z}_1 - z_1(t))^2 + (\tilde{z}_2 - z_2(t))^2 \leq R^2(t)\}, \\ C(t) &= \{(\tilde{z}_1,\tilde{z}_2) : (\tilde{z}_1 - z_1(t))^2 + (\tilde{z}_2 - z_2(t))^2 = R^2(t)\} \end{aligned} \tag{18}$$

where the radius $R(t)$ may be time dependent. Since the sensing regions are often assumed to be circular one can assume that the function $f(\cdot,\cdot,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ has the following form:

$$f(t,\tilde{z}_1,\tilde{z}_2) = g(t,(\tilde{z}_1 - z_1(t))^2 + (\tilde{z}_2 - z_2(t))^2) \tag{19}$$

for some nonnegative and continuously differentiable function $g(\cdot,\cdot) : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$. These assumptions would correspond to the case of an agent having a circular sensing region where the quality of sensing depends on how far the point $(\tilde{z}_1,\tilde{z}_2) \in \mathbb{R}^2$ is away from $(z_1(t), z_2(t))$ which is the position of the agent at time t. Then equation (17) can be rewritten as

$$I(t) = \oint\limits_{C(t)} g(t,(\tilde{z}_1 - z_1(t))^2 + (\tilde{z}_2 - z_2(t))^2)\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right). \tag{20}$$

To simplify the expression in equation (20) let us introduce a change of variables as [42]

$$\begin{aligned} \tilde{z}_1 &= z_1(t) + R(t)\cos\varphi, \\ \tilde{z}_2 &= z_2(t) + R(t)\sin\varphi \end{aligned} \tag{21}$$

which implies

$$g(t,(\tilde{z}_1-z_1(t))^2+(\tilde{z}_2-z_2(t))^2)=g(t,R^2(t)) \text{ on } C(t) \tag{22}$$

and [21]

$$\begin{aligned}\frac{\partial \tilde{z}_1}{\partial t}&=\dot{z_1}(t)+\dot{R}(t)\cos\varphi,\ d\tilde{z}_1=-R(t)\sin\varphi d\varphi,\\ \frac{\partial \tilde{z}_2}{\partial t}&=\dot{z_2}(t)+\dot{R}(t)\sin\varphi,\ d\tilde{z}_2=R(t)\cos\varphi d\varphi.\end{aligned} \tag{23}$$

Then in [42] it was shown that $I(t)=2\pi\dot{R}(t)R(t)g(t,R^2(t))$ which results in

$$\frac{dJ(t)}{dt}=\iint\limits_{D(t)}\frac{\partial f(t,\tilde{z}_1,\tilde{z}_2)}{\partial t}d\tilde{z}_1 d\tilde{z}_2+2\pi\dot{R}(t)R(t)g(t,R^2(t)). \tag{24}$$

This result is interesting in the sense that it shows that the integral $I(t)$ is zero either if the function $f(t,\cdot)$ (that is, $g(t,\cdot)$) is equal to zero on $C(t)$ or the radius $R(t)$ does not depend on time. Also if the radius $R(t)$ is decreasing with time then $I(t)$ is negative which is desirable if a designer uses a Liapunov approach in coverage control where the goal would be to guarantee that dJ/dt is negative if the full coverage is not achieved (like in [15, 16]). Unfortunately this result applies only to the case when the agents' sensing regions do not overlap. In the case when the agents' sensing regions do overlap, the boundary of an area integral is not smooth anymore which makes the computation of the derivative of the area integral (16) more difficult. In order to show how the differentiation should be done in the case of overlapping sensing regions let us start with a scenario with two agents with overlapping sensing domains as depicted in Figure 1. The agents i and j two dimensional position state vectors (usually Cartesian coordinates) at time instant t are denoted as $x_i^p(t)$ and $x_j^p(t)$, respectively. Their sensing radii are denoted as $R_i(t)$ and $R_j(t)$. Now the area of integration $D(t)$ is still a compact region in $\mathbb{R}^2$ yet it is bounded by $C(t)$ which is now only piecewise smooth. In this case $C(t)$ can be represented as the union of two smooth arcs connected at points $A(t)$ and $B(t)$ as depicted in Figure 1. Now let us again consider the problem of differentiating the area integral in equation (15), that is,

$$\frac{dJ(t)}{dt}=\frac{d}{dt}\left(\iint\limits_{D(t)}f(t,\tilde{z}_1,\tilde{z}_2)d\tilde{z}_1 d\tilde{z}_2\right). \tag{25}$$

Furthermore let us assume that we have at least three times continuously differentiable function $F(t,\tilde{z}_1,\tilde{z}_2)$ and recall that the Green's theorem is valid for domains with piecewise smooth boundaries [21, 47] so we can write

$$\iint\limits_{D(t)}\frac{\partial F}{\partial \tilde{z}_2}d\tilde{z}_1 d\tilde{z}_2=-\oint\limits_{C(t)}F d\tilde{z}_1 \tag{26}$$

for any given t. The same result applies to $\partial F/\partial t$, that is

$$\iint\limits_{D(t)}\frac{\partial^2 F}{\partial t\partial \tilde{z}_2}d\tilde{z}_1 d\tilde{z}_2=-\oint\limits_{C(t)}\frac{\partial F}{\partial t}d\tilde{z}_1 \tag{27}$$

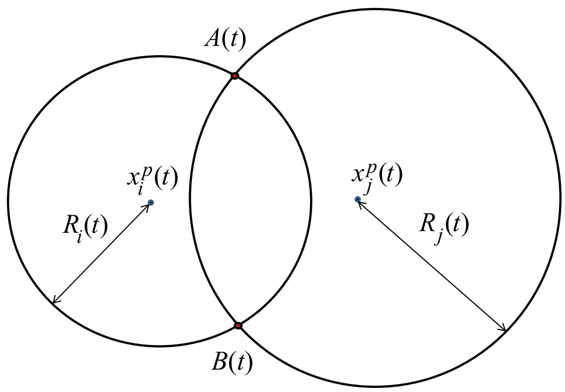

FIGURE 1. Two agents with overlapping sensor domains.

where we use simplified notation for $F = F(t, \tilde{z}_1, \tilde{z}_2)$. At this point we use the formula for the differentiation of a line integral over a smooth arc $\widehat{L(t)} = \widehat{A(t)B(t)}$, with its end points $A(t)$ and $B(t)$, as [21]

$$\frac{d}{dt}\left(\int\limits_{\widehat{L(t)}} F d\tilde{z}_1\right) = \int\limits_{\widehat{L(t)}} \frac{\partial F}{\partial t} d\tilde{z}_1 - \int\limits_{\widehat{L(t)}} \frac{\partial F}{\partial \tilde{z}_2}\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) + F\frac{d\tilde{z}_1}{dt}\Bigg|_{A(t)}^{B(t)}. \tag{28}$$

By differentiating both sides of equation (26) we obtain

$$\frac{d}{dt}\left(\iint\limits_{D(t)} \frac{\partial F}{\partial \tilde{z}_2} d\tilde{z}_1 d\tilde{z}_2\right) = -\frac{d}{dt}\left(\oint\limits_{C(t)} F d\tilde{z}_1\right) \tag{29}$$

and by splitting the integration over $C(t)$ to integration over two smooth arcs $\widehat{A(t)B(t)}$ and $\widehat{B(t)A(t)}$ (integration is done in the counterclockwise or positive math direction) we obtain the following

$$\frac{d}{dt}\left(\iint\limits_{D(t)} \frac{\partial F}{\partial \tilde{z}_2} d\tilde{z}_1 d\tilde{z}_2\right) = -\frac{d}{dt}\left(\int\limits_{\widehat{A(t)B(t)}} F d\tilde{z}_1\right) - \frac{d}{dt}\left(\int\limits_{\widehat{B(t)A(t)}} F d\tilde{z}_1\right). \tag{30}$$

Now from equations (28) and (30) it follows that

$$\begin{aligned}\frac{d}{dt}\left(\iint\limits_{D(t)} \frac{\partial F}{\partial \tilde{z}_2} d\tilde{z}_1 d\tilde{z}_2\right) &= -\oint\limits_{C(t)} \frac{\partial F}{\partial t} d\tilde{z}_1 + \oint\limits_{C(t)} \frac{\partial F}{\partial \tilde{z}_2}\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \\ &\quad + F\frac{d\tilde{z}_1^+}{dt}\Bigg|_{A(t)}^{B(t)} + F\frac{d\tilde{z}_1^-}{dt}\Bigg|_{B(t)}^{A(t)}\end{aligned} \tag{31}$$

where $d\tilde{z}_1^+/dt$ denotes a derivative with respect to a parametrization of a smooth arc $\widehat{A(t)B(t)}$ and $d\tilde{z}_1^-/dt$ denotes a derivative with respect to a parametrization of a smooth arc $\widehat{B(t)A(t)}$. Using equation (27) we can rewrite (31) as

$$\begin{aligned} \frac{d}{dt}\left(\iint\limits_{D(t)} \frac{\partial F}{\partial \tilde{z}_2} d\tilde{z}_1 d\tilde{z}_2\right) &= \iint\limits_{D(t)} \frac{\partial^2 F}{\partial t \partial \tilde{z}_2} d\tilde{z}_1 d\tilde{z}_2 + \oint\limits_{C(t)} \frac{\partial F}{\partial \tilde{z}_2}\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \\ &\quad + F\frac{d\tilde{z}_1^+}{dt}\bigg|_{A(t)}^{B(t)} + F\frac{d\tilde{z}_1^-}{dt}\bigg|_{B(t)}^{A(t)}. \end{aligned} \tag{32}$$

Following the proof of formula (28) provided in [21] we set $\partial F(t, \tilde{z}_1, \tilde{z}_2)/\partial \tilde{z}_2 = f(t, \tilde{z}_1, \tilde{z}_2)$ (which is admissible due to the assumption that $F(\cdot)$ is three times continuously differentiable) and finally obtain:

$$\begin{aligned} \frac{d}{dt}\left(\iint\limits_{D(t)} f(t, \tilde{z}_1, \tilde{z}_2) d\tilde{z}_1 d\tilde{z}_2\right) &= \iint\limits_{D(t)} \frac{\partial f}{\partial t} d\tilde{z}_1 d\tilde{z}_2 \\ &\quad + \oint\limits_{C(t)} f(t, \tilde{z}_1, \tilde{z}_2)\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \\ &\quad + F\frac{d\tilde{z}_1^+}{dt}\bigg|_{A(t)}^{B(t)} + F\frac{d\tilde{z}_1^-}{dt}\bigg|_{B(t)}^{A(t)}. \end{aligned} \tag{33}$$

Equation (33) can be rewritten as

$$\begin{aligned} \frac{d}{dt}\left(\iint\limits_{D(t)} f(t, \tilde{z}_1, \tilde{z}_2) d\tilde{z}_1 d\tilde{z}_2\right) &= \iint\limits_{D(t)} \frac{\partial f}{\partial t} d\tilde{z}_1 d\tilde{z}_2 \\ &\quad + \oint\limits_{C(t)} f(t, \tilde{z}_1, \tilde{z}_2)\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \\ &\quad + F\left(\frac{d\tilde{z}_1^+}{dt} - \frac{d\tilde{z}_1^-}{dt}\right)\bigg|_{B(t)} \\ &\quad + F\left(\frac{d\tilde{z}_1^-}{dt} - \frac{d\tilde{z}_1^+}{dt}\right)\bigg|_{A(t)}. \end{aligned} \tag{34}$$

If the boundary $C(t)$ is piecewise smooth then $d\tilde{z}_1^+/dt$ and $d\tilde{z}_1^-/dt$ derivatives will not match at the points of discontinuity $A(t)$ and $B(t)$. Also, it can be easily deduced from our derivation that if there is an overlap of an arbitrary number of agents's sensing areas with M critical points $A_i(t)$, $i \in \{1, \ldots, M\}$, equation (34) will generalize to

$$\begin{aligned} \frac{d}{dt}\left(\iint\limits_{D(t)} f(t, \tilde{z}_1, \tilde{z}_2) d\tilde{z}_1 d\tilde{z}_2\right) &= \iint\limits_{D(t)} \frac{\partial f}{\partial t} d\tilde{z}_1 d\tilde{z}_2 \\ &\quad + \oint\limits_{C(t)} f(t, \tilde{z}_1, \tilde{z}_2)\left(\frac{\partial \tilde{z}_1}{\partial t} d\tilde{z}_2 - \frac{\partial \tilde{z}_2}{\partial t} d\tilde{z}_1\right) \\ &\quad + \sum_{i=1}^{N} (-1)^{b_i(t)} F\left(\frac{d\tilde{z}_1^{i+}}{dt} - \frac{d\tilde{z}_1^{i-}}{dt}\right)\bigg|_{A_i(t)} \end{aligned} \tag{35}$$

where binary numbers $b_i(t) \in \{0,1\}$ depend on the location of critical points, and $d\tilde{z}_1^{i+}/dt$ and $d\tilde{z}_1^{i-}/dt$ are the derivatives with respect to the parameterizations of the two smooth curves meeting at the point $A_i(t)$.

3.2. **Coverage error function and control laws.** In this subsection we concentrate on a particular design of control laws for nonlinear systems affine in control. Sensing capabilities of an agent i are modeled by the following function [15, 16]:

$$S_i(p) = \frac{M_i}{R_i^4}\max\{0, R_i^2 - p\}^2 \Rightarrow S_i'(p) = -2\frac{M_i}{R_i^4}\max\{0, R_i^2 - p\} \tag{36}$$

and given that the number of agents is N, a cumulative sensing function is given by

$$Q(t,\tilde{x}) = \int_0^t \left(\sum_{i=1}^{N} S_i(\|x_i^p(\tau) - \tilde{x}\|^2)\right) d\tau \tag{37}$$

where $\tilde{x} = [\tilde{x}_1, \tilde{x}_2]^T \in \mathbb{R}^2$ and $x_i^p(\cdot)$ is a two dimensional function of time that represents the planar position of the agent i. Let us define $h(w) = (\max\{0,w\})^3$, so that $h'(w) = 3(\max\{0,w\})^2$ and $h''(w) = 6\max\{0,w\}$. We consider the following coverage error function [15]

$$e(t) = \iint_{\mathcal{D}} h(C^* - Q(t,\tilde{x}))\phi(\tilde{x})d\tilde{x}_1 d\tilde{x}_2 \tag{38}$$

where $\mathcal{D}$ is a given compact domain to be covered, C^* is a positive constant which value is a design parameter and $\phi(\tilde{x})$ is a nonnegative scalar function which can be used to incorporate any preferences or prior information in covering the domain. A value of the parameter C^* is chosen depending on how well we would like to search the area [15, 16]. Larger values of C^* force agents to spend more time sensing each point in the domain and the value itself is what mathematically captures what we denote as the *satisfactory coverage*. It is also interesting to point out that $Q(\cdot)$ and $e(\cdot)$ depend on $x_i^p(\tau)$, $\tau \in [0,t]$, $i \in \mathbf{N}$, yet they are defined as functions of time by assuming that the agents trajectories are known. Also notice that as initial conditions we need only initial values of the state vectors and not vector functions. These facts simplify our notation and allow us to treat $e(\cdot)$ as a function of time. Since our approach is based on the Liapunov analysis we are faced with the issues of computing the time derivative of the error function given in equation (38). Two major issues with this approach can be pointed out by using equation (35). The first issue is the computation of the integral over a boundary $C(t)$ as well as the jumps in equation (35). This issue is bypassed due to the fact that if there is no overlapping between agents' sensing regions these terms are equal to zero and in the case when an overlapping exists our numerical simulations show that its influence is minor and thus can be neglected. The other issue is that the control laws do not appear in the first derivative of the error function which means that we are dealing with the case of singular control [45]. The solutions to problems of computing singular controls are known to be particular and thus highly depend on the formulations of the particular problems. In [15, 16, 42] a function of time $\hat{e}(t)$ defined as

$$\begin{aligned}\hat{e}(t) &= \iint_{\mathcal{D}} \frac{d}{dt}\left(h(C^* - Q(t,\tilde{x}))\phi(\tilde{x})\right) d\tilde{x}_1 d\tilde{x}_2 \\ &= -\iint_{\mathcal{D}} h'(C^* - Q(t,\tilde{x}))\left(\sum_{i=1}^{N} S_i(\|x_i^p(t)-\tilde{x}\|^2)\right)\phi(\tilde{x})d\tilde{x}_1 d\tilde{x}_2 \qquad (39)\end{aligned}$$

which represents one term of the time derivative of $e(t)$, was considered. Again, the time derivative of $e(t)$ includes also a one dimensional integral over the boundary of a domain of integration as well as computing values of jumps as in equation (35). Because this is the case of singular control, it was shown how $\hat{e}(t)$ can be used instead of $e(t)$ in [15, 16, 42]. This was done under the assumption that the agents do not leave the search domain which may allow $\hat{e}(t)$ to converge to zero without implying that $e(t)$ converges to zero. In order to prevent this from happening and without adding an additional constraint, we define a new function of time $\tilde{e}(t)$ as follows:

$$\tilde{e}(t) = -\iint_{\mathcal{D}} h'(C^* - Q(t,\tilde{x}))\left(S^* - \sum_{i=1}^{N} S_i(\|x_i^p(t)-\tilde{x}\|^2)\right)\phi(\tilde{x})d\tilde{x}_1 d\tilde{x}_2 \qquad (40)$$

where S^* is a positive constant that satisfies $S^* > \sum_{i=1}^N M_i$ so that $S^* - \sum_{i=1}^N S_i(p_i)$ > 0 for any $\{p_1, p_2, \ldots, p_N\}$. The motivation to use $\tilde{e}(t)$ instead of $\hat{e}(t)$ is that the convergence of $\tilde{e}(t)$ to zero implies that $e(t)$ converges to zero yet $\hat{e}(t)$ can converge to zero by forcing $\sum_{i=1}^N S_i(p_i)$ to go to zero by, for example, agents just leaving the search area which will not result in $e(t)$ converging to zero. Now we proceed by defining

$$\begin{aligned}\tilde{\tilde{e}}(t) &= -\iint_{\mathcal{D}} \frac{d}{dt}\left(h'(C^* - Q(t,\tilde{x}))(S^* - S(t,\tilde{x}))\phi(\tilde{x})\right) d\tilde{x}_1 d\tilde{x}_2 \\ &= \iint_{\mathcal{D}} h''(C^* - Q(t,\tilde{x}))S(t,\tilde{x})(S^* - S(t,\tilde{x}))\phi(\tilde{x})d\tilde{x}_1 d\tilde{x}_2 \\ &\quad +2\sum_{i=1}^{N}\iint_{\mathcal{D}} h'(C^* - Q(t,\tilde{x}))S_i'(p_i(t,\tilde{x}))(x_i^p(t)-\tilde{x})^T \dot{x}_i^p(t) d\tilde{x}_1 d\tilde{x}_2 \qquad (41)\end{aligned}$$

where $p_i(t,\tilde{x}) = \|x_i^p(t) - \tilde{x}\|^2$ and $S(t,\tilde{x}) = \sum_{i=1}^N S_i(p_i(t,\tilde{x}))$. Equation (41) can be rewritten as

$$\tilde{\tilde{e}}(t) = a_0(t) + \sum_{i=1}^{N}(a_{i1}(t)\dot{x}_{i1}^p(t) + a_{i2}(t)\dot{x}_{i2}^p(t)) = a_0(t) + \sum_{i=1}^{N} a_i^T(t)\dot{x}_i^p(t) \qquad (42)$$

where $\dot{x}_i^p(t) = [\dot{x}_{i1}^p(t), \dot{x}_{i2}^p(t)]^T$, $a_i(t) = [a_{i1}(t), a_{i2}(t)]^T$, and

$$
\begin{aligned}
a_0(t) &= \iint_{\mathcal{D}} h''(C^* - Q(t,\tilde{x}))S(t,\tilde{x})(S^* - S(t,\tilde{x}))\phi(\tilde{x})d\tilde{x}_1 d\tilde{x}_2, \\
a_{i1}(t) &= 2\iint_{\mathcal{D}} h'(C^* - Q(t,\tilde{x}))S_i'(p_i(t,\tilde{x}))(x_{i1}^p(t) - \tilde{x}_1)d\tilde{x}_1 d\tilde{x}_2, \\
a_{i2}(t) &= 2\iint_{\mathcal{D}} h'(C^* - Q(t,\tilde{x}))S_i'(p_i(t,\tilde{x}))(x_{i2}^p(t) - \tilde{x}_2)d\tilde{x}_1 d\tilde{x}_2.
\end{aligned} \tag{43}
$$

Now, if we assume that the agents' partial dynamics corresponding to the position variables are given by $\dot{x}_i^p = g_i^p(x_i)u_i^c$ then u_i^c is only a portion of the agent's control vector which corresponds to the position state variables. This portion of the control vector may be designed as

$$
u_i^c(t) = k_i^c g_i^p(x_i)^T a_i(t),\ i \in \mathbf{N} \tag{44}
$$

where k_i^c is a positive coverage gain for an agent i. It is interesting to note that the difference between this control law and the ones used in [15, 16, 42] is the sign which is due to the assumption that the agents are not assumed to leave the search domain. The proposed control laws result in

$$
\dot{\tilde{e}}(t) = a_0(t) + \sum_{i=1}^{N} k_i^c \left\| g_i^p(x_i)^T a_i(t) \right\|^2 . \tag{45}
$$

Notice that $\tilde{e}(t)$ in equation (40) is nonpositive by construction which implies that $\dot{\tilde{e}}(t)$ needs to be nonnegative. If the agents' dynamics include extra terms $h_i(\cdot)$, $i \in \mathbf{N}$, as formulated in equation (1), the control laws would stay the same yet it would result in additional terms appearing in equation (45). This would require further investigation of the nonnegativity of $\dot{\tilde{e}}(t)$ depending on particular functions $h_i(\cdot)$, $i \in \mathbf{N}$.

3.3. **A numerical example.** To illustrate the methodology for computing control laws for agents described by nonlinear yet affine in control models, we assume that the models are unicycles, that is, differential drives. Therefore we assume that each agent i, where the total number of agents is still denoted by N, that is, $i \in \mathbf{N} = \{1, \ldots, N\}$, is modeled using the following unicycle model:

$$
\begin{aligned}
\dot{x}_{i1}^p &= u_{i1}\cos(x_{i3}) \\
\dot{x}_{i2}^p &= u_{i1}\sin(x_{i3}) \\
\dot{x}_{i3} &= u_{i2}
\end{aligned} \tag{46}
$$

where $x_i^p = [x_{i1}^p, x_{i2}^p]^T$ denotes position state variables, x_{i3} is an orientation angle, and $x_i = [(x_i^p)^T, x_{i3}]^T$ is the state vector. Speed control input is denoted as u_{i1} and angular velocity control as u_{i2} so that the overall control input for the i-th agent is given by $u_i = [u_{i1}, u_{i2}]^T$. Thus the unicycle dynamic model fits the models in equation (1) with:

$$
g_i(x_i) = \begin{bmatrix} \cos(x_{i3}) & 0 \\ \sin(x_{i3}) & 0 \\ 0 & 1 \end{bmatrix}. \tag{47}
$$

At this point it is important to note that the three objectives are described by functions given in equations (2), (3), (5), and (40) which are functions of position variables. As a consequence objective Liapunov-like functions depend only on the position state variables, that is, a Liapunov-like function for the i-th agent can be represented as

$$v_i \equiv v_i(x_i^p, \bar{x}_i^p), \ \bar{x}_i^p = \{x_j^p : j \neq i, \ j \in \mathcal{N}_i\} \tag{48}$$

where $\mathcal{N}_i$ is a *set of agents interacting* with agent i. This set is defined by arguments of $v_i(\cdot)$. Then, the control laws of the agents, given by equations (12) and (44), aim to decrease (in the case of collision avoidance and proximity objectives) or increase (coverage objective) functions that can be represented in the following form:

$$w_i(t, x, u_i) = (b_{i1}(t,x)\cos(x_{i3}) + b_{i2}(t,x)\sin(x_{i3}))\, u_{i1} \tag{49}$$

where $b_{ij}(t,x) = \partial v_i / \partial x_{ij}, \ j \in \{1,2\}$, for the collision avoidance and proximity control laws ($v_i(\cdot)$ is given in equation (11)) and $b_{ij}(t,x) = a_{ij}(t)$, $j \in \{1,2\}$, for the coverage control laws (coefficients $a_{ij}(t)$ are given in equation (43)). This implies that we can choose control laws u_{i1} as

$$u_{i1}(t,x) = c_i k_i^s \left(b_{i1}(t,x)\cos(x_{i3}) + b_{i2}(t,x)\sin(x_{i3})\right), \ i \in \mathbf{N} \tag{50}$$

where $c_i = -1$ for collision avoidance and proximity objectives and $c_i = 1$ for the coverage objective. Positive gain k_i^s is related to speed and is a design parameter. By substituting equation (50) into equation (49) we obtain

$$\begin{aligned} w_i(t,x,u_i(t,x)) &= c_i k_i^s \left(b_{i1}\cos(x_{i3}) + b_{i2}\sin(x_{i3})\right)^2 \\ &= c_i k_i^s (b_{i1}^2 + b_{i2}^2)\sin^2\left(x_{i3} + \arctan\left(\frac{b_{i1}}{b_{i2}}\right)\right) \end{aligned} \tag{51}$$

where for short notation we denote $b_{ij}(t,x)$ as b_{ij}, $j \in \{1,2\}$. From equation (51) it can be concluded that the control u_{i1} is maximally efficient (in either case) when the heading angle equals to

$$x_{i3}^o = \varphi_i^o = \pi/2 - \arctan\left(\frac{b_{i1}}{b_{i2}}\right). \tag{52}$$

Now, the angular velocity control of an i-th agent which would steer the agent towards the desired heading angle can be designed as a PD controller of the form [26]

$$u_{i2} = -k_i^\omega \left(x_{i3} - \varphi_i^o\right) + \frac{d\varphi_i^o}{dt} \tag{53}$$

where k_i^ω denotes a positive gain which is another design parameter.

Now let us consider a scenario with three nonhomogeneous unicycles and two obstacles. To simplify the presentation we denote agents with indices $\{1,2,3\}$ and obstacles with indices $\{4,5\}$. One of the objectives for the three agents is to search a 32×32 square domain placed in the positive quadrant with its lower left vertex positioned at the origin. The initial positions for the agents 1, 2, and 3 are at $(12,7)$, $(14,15)$ and $(24,7)$, respectively. All the units are assumed to be normalized and thus need not be specified. Two obstacles denoted by indices 4 and 5 are of ellipsoidal shapes centered at $(7,12)$ and $(16,25)$, respectively. Positive definite matrices for avoidance functions (given in equations (2) and (3)) which are not equal to the 2×2 identity matrix are $P_{14}{=}P_{15}{=}P_{24}{=}P_{25}{=}P_{34}{=}P_{35}{=}\mathrm{diag}\{0.5,1\}$.

Avoidance and detection regions are spherical regions specified by $r_{1j} = 1$, $R_{1j} = 3$, $r_{2j} = 1$, $R_{2j} = 3$, $r_{3j} = 1$, $R_{3j} = 3$. It is assumed that the ellipsoids defined by P_{ij} and R_{ij}, $i \in \{1, 2, 3\}$, $j \in \{4, 5\}$, define the two obstacles 4 and 5, respectively. The proximity and avoidance control gains which appear in equation (12) and refer to the speed control input are $\hat{k}_1 = 1$, $\hat{k}_2 = 1$, and $\hat{k}_3 = 1$. Similar to the numerical example scenario considered in [42], we assume that only agents 2 and 3 are in charge of avoiding collisions between the agents yet all three agents need to avoid the obstacles. Unlike the scenario considered in [42] we assume that all agents are trying to satisfy the proximity constraints. The proximity regions are the same and defined by $\hat{R}_{ij} = 10$ for all i and j. The scaling coefficients (in equation (11)) for the collision avoidance objective functions are chosen as $\gamma_{1j} = .015$, $\gamma_{2j} = .0015$, and $\gamma_{3j} = .0015$ for all j. The scaling coefficients (again we refer to equation (11)) for the proximity objective functions are chosen as $\gamma_{1j} = .0001$, $\gamma_{2j} = 3.5 \cdot 10^{-5}$, and $\gamma_{3j} = 10^{-5}$ for all j. Also, $\delta = 2$ in equation (7).

Agents are also assumed to be nonhomogeneous in terms of their sensing capacities and their coverage regions' radii are $R_1 = 32\sqrt{2}$, $R_2 = 2$, $R_3 = 2$, and their maximal peak sensing capacities are $M_1 = 1/75$, $M_2 = 6$, and $M_3 = 6$, respectively. Again, similar to the scenario proposed in [42] we choose a coverage/sensing region for agent 1 to be capable to sense the whole domain from any position in the domain and in this way avoid using additional global coverage control as proposed and needed in [15, 16]. Agents' speed coverage control gains in equation (44) are $k_1^c = 2.5 \cdot 10^{-4}$, $k_2^c = 5 \cdot 10^{-6}$, $k_3^c = 2.5 \cdot 10^{-5}$, and the coverage constant is $C^* = 40$ while the prior knowledge function $\phi(\cdot)$ is set to be identically equal to one. It is assumed that any two agents i and j exchange coverage information only if their relative distance is smaller than $\hat{R}_{ij} = 10$. This is incorporated into our analysis by modifying the summation over all agents with a summation over a set of agents that can exchange the information in equations (37)-(40). The overall speed control law for each agent was obtained as the sum of the coverage speed control law and combined avoidance and proximity speed control law. The heading angle control law for each agent was designed by using only a proportional part of the PD control law given in equation (53) with gains set as $k_i^\omega = 1$, $i \in \{1, 2, 3\}$, where the desired angle is equal to the sum of the coverage desired angle and combined avoidance and proximity desired angle, each being computed using equation (52) with appropriate corresponding coefficients $b_{ij}(\cdot)$.

The top view of agents' trajectories as well as the two obstacles is provided in Figure 2. Agents' initial positions are depicted as green dots and agents' final positions as red dots. Trajectories for agents 1, 2 and 3 are depicted in magenta, black and blue, respectively. The coverage error function (38) versus time is given in Figure 3. A color coded depiction of the quality of coverage with agents' final positions is provided in Figure 4 where the satisfactory coverage is chosen as $C^* =$ 40 and color coded in light green. Finally pairwise distances between the agents themselves as well as agents and obstacles together with avoidance, detection and proximity levels are given in Figure 5. The coverage error function converged to zero in $T = 7919$[s].

Conclusions. In this paper we proposed an approach to avoidance, coverage and proximity control based on the concept of avoidance control and careful constructions of individual objectives' functions. Agents' dynamic models are assumed to be

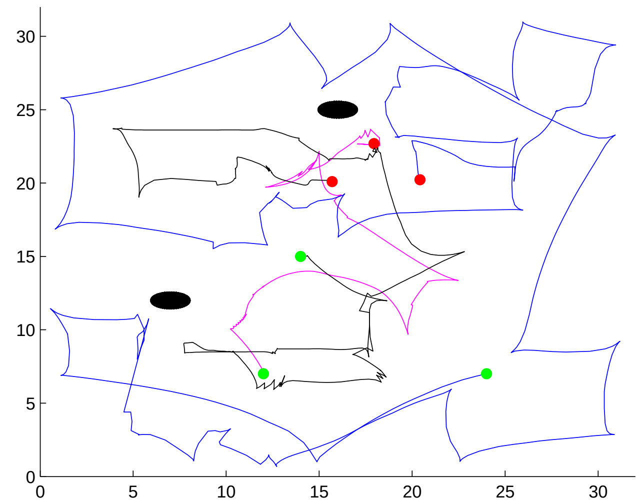

FIGURE 2. Agents' trajectories and obstacles.

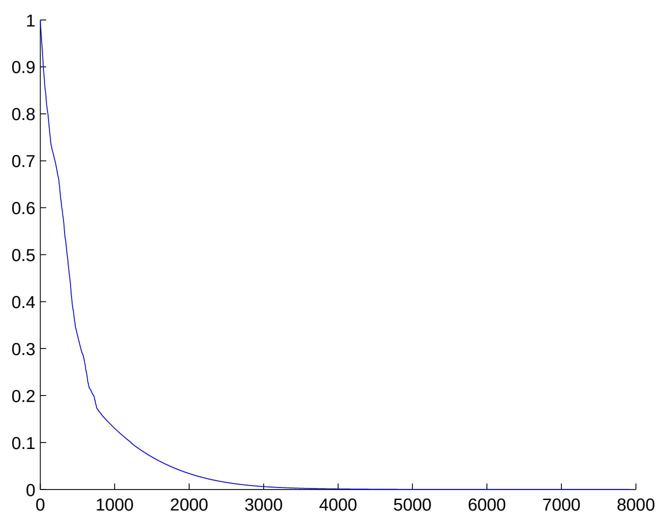

FIGURE 3. The coverage error function versus time.

nonlinear yet affine in controls. We also indicate and provide some solutions to coverage control related problems which stem from the differentiation of area integrals depending on time as a parameter. As an illustration, a numerical example with three nonhomogeneous nonholonomic agents and two static obstacles was provided.

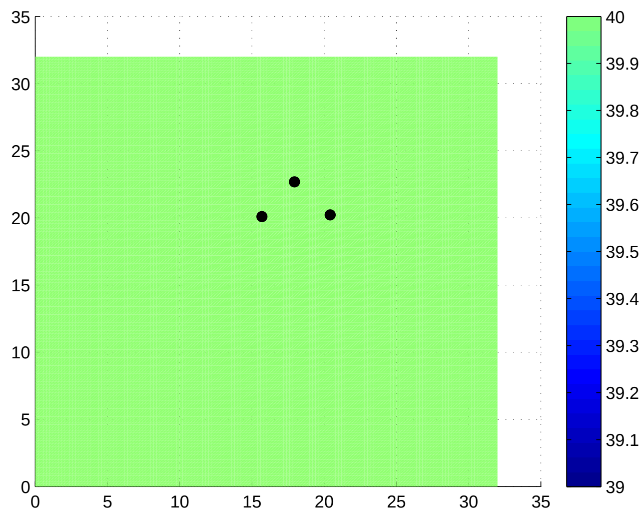

FIGURE 4. Color coded quality of coverage with agents' final positions.

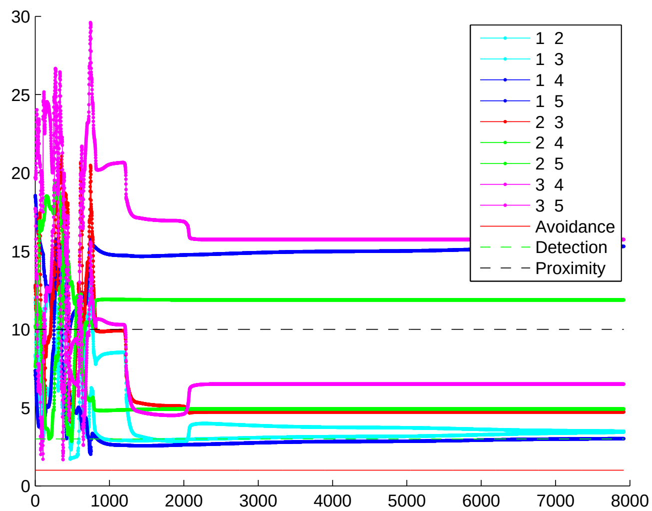

FIGURE 5. Pairwise distances between agents, and agents and obstacles.

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