State estimation in wall-bounded flow systems. Part 3. The Ensemble Kalman Filter

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State estimation of turbulent near-wall flows based on wall measurements is one of the key pacing items in model-based flow control, with low-Re channel flow providing the canonical testbed. Model-based control formulations in such settings are often separated into two subproblems: estimation of the near-wall flow state via skin friction and pressure measurements at the wall, and (based on this estimate) control of the near-wall flowfield fluctuations via actuation of the fluid velocity at the wall. In our experience, the turbulent state estimation subproblem has consistently proven to be the more difficult of the two. Though many estimation strategies have been tested on this problem (by our group and others), none have accurately captured the turbulent flow state at about 20 viscous units from the walls, which is deemed to be an important milestone, as this is the approximate location of the characteristic near-wall turbulent structures. Leveraging the Ensemble Kalman Filter (an effective variant of the Kalman filter developed in the weather forecasting community which scales well to high-dimensional systems), the present paper achieves an order of magnitude improvement (in the near-wall region) over the best results available in the published literature on the estimation of low-Reynolds number turbulent channel flow, based on wall information alone. Additionally, when starting from a good (but not perfect) initial estimate and using the appropriate localization parameters, the estimator actually essentially synchronizes with the entire turbulent flow in the channel, and maintains this near-perfect synchronization, apparently indefinitely.

1. Introduction

State estimation is a problem that has been considered at length by researchers in many distinct communities. The “controls” and “dynamic systems” communities have focused primarily, but not exclusively, on problems which

(a) are of sufficiently low state dimension (or, at least, problems that can effectively be reduced to such via standard model reduction techniques) that the corresponding Riccati equations, LMIs, or dynamic programs are numerically tractable, and

(b) are characterized by uncertainties in the initial state, state disturbances, and measurement noise that are well approximated as Gaussian.

In most cases, these assumptions are not valid in estimation problems related to turbulent flows.

The weather forecasting community, on the other hand, has focused on estimation (a.k.a. “data assimilation”) strategies that are numerically tractable for high-dimensional discretizations of PDE systems. The two primary classes of data assimilation strategies which have been developed in this community and are available today for multiscale uncertain systems are:

• the Ensemble Kalman Filter (EnKF; see Evensen, 2003), and

• space/time variational (4DVar; see Bouttier and Courtier, 1999) methods.

EnKF methods, which come in a few distinct variations, are particularly well suited for non-linear multiscale systems with substantial uncertainties. Even for some low-dimensional problems,
EnKF methods have been shown to provide significantly improved state estimates in certain nonlinear problems for which the more traditional Extended Kalman Filter breaks down. The statistics of the estimation error in the EnKF are not propagated via a covariance matrix, but instead are implicitly represented via the distribution of several perturbed trajectories ("ensemble members"), which themselves are propagated with the full nonlinear system model. On many problems, in practice, the collection of these ensemble members (itself called the "ensemble") accurately captures the dominant directions of uncertainty of the estimation error (in phase space) even when a relatively small number of ensemble members is used; this is the key feature that lends the EnKF method its remarkable numerical tractability in high-dimensional problems.

4DVar methods, on the other hand, propagate state and sensitivity ("adjoint") simulations back and forth across an optimization window of interest. An optimization is performed based on these iterative marches in order to minimize a cost function balancing: (a) a term accounting for the misfit of the estimate with the measurements over the optimization window, with (b) a "background" term accounting for the "old" estimate (that is, based on the measurements and statistics obtained prior to the present optimization window). Though such a retrospective analysis is certainly beneficial in certain ways in the estimation of nonlinear systems, 4DVar methods are not as natural as EnKF methods for representing the principle directions of uncertainty in the estimate, which is a crucial ingredient of any effective state estimation strategy.

1.1. Prior work on the estimation of turbulent channel flow

In the past, estimation of chaotic fluid systems was mostly motivated by the need for accurate weather forecasts. Today, the prospects of potential implementation of real-time feedback control in manufacturing systems, or perhaps even aerodynamics systems, provide new motivation to study this fundamental problem. Bewley & Liu (1998), Bewley (2001) and Hörgberg et. al. (2003) developed optimal feedback kernels for the control of the linearized Navier-Stokes equations in a channel. The dependence of these feedback control kernels on the near-wall region, where the coherent structures are present, emphasizes the importance of accurate state estimates in this region if effective feedback control is the ultimate aim. Bewley and Protas (2004) attempted to use 4DVar to estimate an $Re = 180$ flow, but the results were not of sufficient accuracy to suggest that feedback control based on this estimate would be successful. In that paper the authors also examined the direct extrapolation of wall skin friction and pressure measurements of a turbulent flow into the flowfield via Taylor series analyses; unfortunately, it was found that the domain of convergence of such Taylor series was very much smaller than 20 viscous units.

Hörgberg et. al. (2003) also attempted to develop the appropriate kernels for the estimation (that is, Kalman filtering) of near-wall flows, but the estimation formulation used did not converge upon grid refinement, and thus led to spurious results. Hrepprner et. al. (2005) corrected this error, and found effective kernels for the state estimation problem in transitional channel flow (that is, for small perturbations from the laminar state). Chevalier et. al. (2006) then attempted to develop a nonlinear extension of this work in order to apply it effectively to a fully-developed turbulent flow using an Extended Kalman Filter. This work meticulously calculated a numerical model of the statistics of the nonlinear terms of a fully-developed channel flow, then used these statistics as the covariance of the state disturbances when computing the estimator feedback gains via the (linear) Kalman formulation. Unfortunately, the effectiveness of this approach was limited at best and, again, the results were not of sufficient accuracy to suggest that feedback control based on this estimate would be successful.

The groundwork described in the previous two paragraphs, upon which the present paper is based, is reviewed further, and put into a broader context, in Kim & Bewley (2006). A novel alternative strategy to the near-wall turbulent flow estimation problem using a Weiner filter approach, again based on a numerical model of the statistics of a turbulent flow, was explored in
Martinelli (2009), albeit, again, with limited effectiveness. The broad range of existing studies on this canonical problem provides a benchmark against which new approaches may be compared.

The present paper applies the Ensemble Kalman Filter (EnKF) to estimate a $Re_T = 100$ turbulent flow based on wall skin friction and wall pressure measurements alone. The remainder of this introduction summarizes briefly the Ensemble Kalman Filter used, and reviews the principle heuristic, localization, required in its application to large-scale systems. The result, presented in §2, is what might be considered as the first “successful” estimation of this difficult benchmark problem.

1.2. Related background on state estimation

The celebrated Kalman Filter (KF; see Kalman, 1960) is used widely for the estimation of linear systems, and constructs an optimal Bayesian estimate in such systems when the (random) state disturbances and measurement errors corrupting the estimate are Gaussian, as the probability density function (PDF) of the state estimate itself is, in this case, also Gaussian. The KF simply propagates the mean (of order $n$) and covariance (of order $n^2$) of this PDF according to the linear model of the system, and updates both appropriately (via Bayes’ rule) whenever measurements are taken. In estimation problems in which the estimation errors can be kept relatively small, nonlinear systems may also be estimated via an easy extension of this approach, dubbed the Extended Kalman Filter (EKF), simply by propagating the mean with the full nonlinear model, and propagating the covariance based on a linearization of this model about the current estimate. Both the KF and EKF are numerically tractable for moderate values of $n$ i.e., $n \lesssim O(10^3)$.

Note that, in certain linear spatially-homogeneous systems, transform methods may be used to decouple large KF problems into many independent smaller KF problems (see, e.g., Bewley & Liu 1998).

In nonlinear estimation problems in which estimation errors are not small, the PDF of the state estimate is not Gaussian, even if the state disturbances and measurement errors corrupting the estimate are, and thus the KF approach described above is ineffective. In this case, the PDF of the estimate in phase space must be discretized and propagated for accurate results. This converts the straightforward propagation of statistics in the KF problem (with two coupled ODEs, of order $n$ and $n^2$) into a much more difficult PDE, of dimension $n$, governing the evolution of the PDF itself; this PDE is known as the Fokker-Plank equation (see, e.g., Jazwinski 1970, p. 164). This PDE may be approximated and evolved with a Lagrangian method, referred to in this setting as a Particle Filter (PF; see Arulampalam 2002), or with a grid-based method, which may be made tractable by exploiting the sparsity of the PDF in phase space (see Bewley & Sharma 2010); both approaches are numerically tractable only for extremely small values of $n$ i.e., $n \lesssim 5$.

In high-dimensional estimation problems relevant for the estimation of turbulent flows [e.g., $n \gtrsim O(10^6)$], the KF and EKF approaches are infeasible due to their reliance on the propagation of the covariance (of order $n^2$) of the PDF of the estimate, and Bayesian methods based on propagating the full Fokker-Plank PDE, such as the PF method, are completely out of the question. One approach to reducing the dimension of covariance propagation equation in the EKF is Chandrasekhar’s method (see Kailath 1973), which involves the propagation of a reduced-order factored form of the time derivative of the covariance matrix, as well as the propagation of the feedback gain matrix itself, rather than the (numerically-intractable) propagation of the full covariance matrix. This approach is promising for problems of this class, and has yet to be tried.

1.3. The Ensemble Kalman Filter

The Ensemble Kalman Filter (EnKF), first proposed by Evensen (1994), is a modern stochastic alternative to Chandrasekhar’s method, described above, for state estimation in high dimensional systems, and is reviewed in depth in Evensen (2003). The EnKF is similar to the PF in that they are both based on the propagation of a set of perturbed candidate realizations of the system through phase space via the governing equations. The PF uses a weighted linear combination of
these perturbed candidate realizations to approximate the PDF of the estimate, with the weights being adjusted each time a measurement is taken via Bayes’ rule. The EnKF, on the other hand, effectively uses identical weights on each realization, instead shifting the realizations themselves, according to a Kalman-like update formula, whenever measurements are taken; Evensen (1994) established that this approach, when formulated correctly, converges to the Kalman result when the system is linear and a sufficiently large number of ensemble members is used. As mentioned in the introduction, it is often found in practice that this ensemble-based approach accurately captures the dominant directions of uncertainty of the estimation error (in phase space) even when a relatively small number of ensemble members is used. Note that, even though only the second-order statistics of the distribution are typically used at each measurement update in the EnKF approach, the full nonlinear dynamics of the system are used to propagate each candidate realization between measurement updates.

We now review briefly the formulation of the standard EnKF, using a continuous-time representation of the system state $x(t)$ and measurements $y_k = y(t_k)$ available at discrete times $t_k$:

$$\frac{\partial x(t)}{\partial t} = f(x(t), u(t), w(t)), \quad (1.1)$$

$$y_k = h(x_k) + v_k. \quad (1.2)$$

The system dynamics $f(\cdot)$ in this formulation may be nonlinear and forced by some known function $u(t)$, and are also assumed to be corrupted by random “state disturbances” $w(t)$ with known statistics. Similarly, the measurement operator $h(\cdot)$ may be nonlinear, and is assumed to be corrupted by random, white, “measurement noise” $v_k$ with covariance $R_k$.

The EKF propagates the full covariance matrix, and uses it to perform measurement updates according to Bayes’ rule, assuming a Gaussian PDF. The EnKF is, in a sense, quite similar, but builds an estimate $P^e$ of the covariance matrix $P$ based on an outer product matrix quantifying the deviation of the ensemble members from their mean:

$$P^e = \frac{(\delta\hat{x})(\delta\hat{x})^H}{N-1}, \quad \delta\hat{x} = [\delta\hat{x}^1, \delta\hat{x}^2, \ldots, \delta\hat{x}^N], \quad \delta\hat{x}^j = \hat{x}^j - \bar{x}, \quad \bar{x} = \frac{1}{N} \sum_j \hat{x}^j. \quad (1.3)$$

Using this “sample” (that is, approximate) covariance matrix $P^e$, a standard KF measurement update may be performed:

$$\hat{s}_{j,k}^i = \hat{s}_{j,k-1}^i + P^e_{k-1} H_k^T (H_k P^e_{k-1} H_k^T + R_k)^{-1} (y_k - H_k \hat{s}_{j,k-1}^i + v_k^i), \quad (1.4)$$

where

- $\hat{s}_{j,k-1}^i$ denotes the $j$’th ensemble member at timestep $k$ based on measurements up to $y_{k-1}$,
- $P^e_{k-1}$ denotes the sample covariance matrix $P^e$, as given in (1.3), based on the collection of ensemble members $\hat{s}_{j,k-1}^i$ for $j = 1, \ldots, N$,
- $v_k^i$ denotes a discrete-time random vector with covariance $R_k$, and
- $H_k$ denotes a linearization of the operator $h(\cdot)$ about the mean state estimate $\bar{x}_{k-1}$.

For more information on the standard KF measurement update and its properties, the reader is referred to Anderson and Moore (1979).

It is important to note that, in the efficient computational implementation of the EnKF, the outer product formula for the sample covariance matrix (1.3) is kept in its factored form when calculating the update (1.4) [or the modified form of this update, given below] in order to retain the numerical tractability of the result. That is, $P^e$ is represented as the product of two matrices of order $n \times N$ and $N \times n$, bypassing the full computation of the $n \times n$ matrix $P^e$, which is important because, in the implementation, $N \ll n$. 


1.4. Localization

Localization is an artificial, ad hoc, distance-based suppression of the off-diagonal components of the sample covariance matrix $P^e$ as calculated in (1.3). It was first proposed by Houtekamer and Mitchell (2001), and is an absolutely essential ingredient to the success of the EnKF in practice. It is introduced to eliminate spurious correlations in the covariance matrix that arise from the fact that it is usually grossly undersampled (that is, in application, $N \ll n$). Note in (1.3) that the off-diagonal components of the covariance matrix $P^e$ are obtained by by averaging the product of a flowfield perturbation at one point in the physical domain with a flowfield perturbation at another point in the physical domain. If these two points are separated by a large distance, it may be argued on physical grounds that this product should approach zero; localization thus imposes this decay of correlation with distance, even if the system is so grossly undersampled that (1.3) does not capture this (which, in application, is usually the case).

The sample covariance matrix $P^e$ in (1.4) may thus be replaced by

$$P^e = \frac{\rho \bullet (\delta \hat{x})(\delta \hat{x})^H}{N - 1}$$

where $\rho$ is a distance-based localization function, and $\bullet$ denotes the element-wise product. Using this modified sample covariance formula, (1.4) may be rewritten as

$$\hat{x}_{i,k}^j = \hat{x}_{i,k}^j + \rho_1 \bullet P_{i,k}^e H_j^H (\rho_2 \bullet (H H_j^H + R)^{-1} (y_k - H \hat{x}_{i,k}^j + \nu^j),$$

where $\rho_1^{im}$ is a distance-based localization function relating the $i$-th state and the $m$-th measurement, and $\rho_2^{m_1, m_2}$ is a localization function relating $m_1$-th measurement and $m_2$-th measurement; both functions approach unity as the distance between the corresponding flow quantities approaches zero, and both approach zero as the distance between the corresponding flow quantities becomes large.

2. Numerical Results

We now characterize the ability of the EnKF, as described above, to estimate a 3D incompressible turbulent channel flow, given measurements of the skin friction and pressure on uniformly-spaced $16 \times 16$ array on each wall. The numerical computations presented use the standard spectral-spectral-second-order-finite-difference code of Bewley et al. (2001) to simulate the uncontrolled, constant mass-flux turbulent channel flow at $Re = 100$ on a $64^3$ grid. In all numerical experiments, the “truth” model is calculated with an identical simulation code running in parallel with the EnKF-based estimator.

Two criteria were used to quantify the quality of the reconstruction: (1) the error norm, and (2) the correlation of the perturbation components of the estimated and actual fields, defined as follows (see Bewley & Protas 2004):

$$\text{Err}(q_{est}, q_{true}) = \frac{\int_0^{L_1} \int_0^{L_3} (q_{est} - q_{true})^2 \, dz \, dx}{\int_0^{L_1} \int_0^{L_3} (q_{true})^2 \, dz \, dx},$$

(2.1)

$$\text{Corr}(q_{est}, q_{true}) = \frac{\int_0^{L_1} \int_0^{L_3} q_{est} q_{true} \, dz \, dx}{\sqrt{\int_0^{L_1} \int_0^{L_3} (q_{est})^2 \, dz \, dx} \sqrt{\int_0^{L_1} \int_0^{L_3} (q_{true})^2 \, dz \, dx}},$$

(2.2)

Note that primed quantities represent the perturbation component of the velocity field (that is, the instantaneous velocity component minus its planar average). The subscripts $(_{est}$ and $(_{true}$ correspond to the “estimated” and “truth” values, respectively. These two normalized measures account for the $(x,z)$-planar-averaged statistics as a function of time and distance from the wall.
The long-time average of these measures provides a rigorous quantification of the quality of the state estimate as a function of distance from the wall, approximating their corresponding expected values, $E[\text{Err}(y,t)]$ and $E[\text{Corr}(y,t)]$, at statistical steady state.

The error norm defined above is perhaps the more sensitive of the two criteria. It is normalized by the planar-averaged mean-squared energy of the truth simulation, which makes it a particularly sensitive measure near the wall, where this quantity approaches zero. Note in particular that an error norm near unity indicates that the estimate is completely decoupled from the truth, whereas an error norm near zero indicates that the estimate is in perfect agreement with the truth. When significant error is present, the correlation is useful to quantify the planar-averaged phase error, as distinct from the planar-averaged amplitude error; note in particular that an error in the amplitude of the estimate (but not its phase) will adversely affect the error norm, but not the correlation. Note also that a correlation near unity indicates perfect phase alignment of the estimate with the truth.

As mentioned in §1.4, localization is an ad hoc yet critical ingredient to the success of any EnKF implementation. The distance-dependent localization functions $\rho$ used in the present work were chosen to be exponential in shape,

$$\rho_{ij} = \rho(x(i), x(j)) = e^{-|x_i - x_j|^2 / Q},$$

where $Q > 0$ is a diagonal weighting matrix related to three length scales. These length scales, in turn, were determined from correlation studies of uncontrolled turbulence, analogous to those reported in Kim et al. (1987) and Bewley et al. (2001), and subjected to a minor amount of additional variation. The computational expense of the EnKF simulations performed, and the desire to drive them all the way to statistical steady state, prevented us from performing more extensive parametric studies on these three length parameters, or repeating the study at higher Reynolds numbers, both of which are left for future work.

Note that, in all cases reported below, after running the EnKF for long enough that the error of the EnKF simulation appeared to reach statistical steady state, the truth and the EnKF estimator were then run for a full twenty additional flow-through times, to ensure that the statistics of the estimator were fully converged, and the error norm and correlation were averaged in time over this extended period. Note that the symmetry across the channel centerline in Figures 1–4 reflect the completeness of the statistical convergence in the simulations performed. This symmetry would be perfect if statistical steady state were in fact reached; we note that it is in fact quite close in all cases reported. Note also that only 66 ensemble members were used in each EnKF simulation reported below.

### 2.1. Tracking, and a parametric study of the localization function

To characterize the dependence of the EnKF on the length scales parameterizing the localization (see §1.4), we first studied the problem of tracking; that is, starting from a good (but not perfect) initial estimate of a fully-developed turbulent flow, how well (as quantified by the error norm and the correlation, defined above) could the estimator track the turbulent flowfield fluctuations of the truth model. The results are reported in Figures 1 and 2, which depict the error norm and correlation, respectively, of four data assimilation cases, which differ only in the parameters defining the localization function used. All four of these calculations were initialized with an ensemble mean equal to the truth, and a distribution of perturbed ensemble members that was Gaussian.

The results indicate that, when the three parameters are tuned appropriately, the EnKF essentially synchronizes with the turbulent flow indefinitely. This result is remarkable; nothing like it has ever been achieved on this canonical problem. Even when the three parameters are tuned such that the EnKF does not synchronize to the turbulent flow, we note that the error norm and correlation of the estimate are still quite good, as discussed further below.
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2.2. State Estimation

A more significant test of the estimator in this problem, of course, is to quantify its convergence starting from a bad initial guess. Figures 3 and 4 report such a test, using the most suitable values of the localization parameters identified thus far, (50,50,25), as determined in §2.1. Initial conditions in this case were generated from a fully-developed turbulent flow simulation that was completely independent of the state of the truth model; snapshots of this flow were taken every $\Delta t = 2$ time units (normalized by $u_\tau$ and the channel half width) in order to initialize each ensemble member.

In this case, synchronization of the EnKF with the truth was not observed. In fact, the error norms and correlations computed are about equivalent with those of the unsynchronized tracking simulations reported in §2.1. However, note that the EnKF estimation was found to perform
with an order of magnitude less error at 20 viscous units from the wall than the previous best estimation results reported in the literature on this problem (see §1.1). In particular, we observe error measurements of 0.04, 0.09, 0.08 and correlation measurements of 0.999, 0.995, and 0.996 (in the $u$, $v$, and $w$ velocity components, respectively). As a point of comparison Chevalier et. al. (2006, when using the Extended Kalman Filter) reported error measurements of 0.5, 0.88 and 0.9 and correlation measurements of 0.87, 0.59, and 0.59 at the same location.

3. Conclusions and future work

The EnKF depends on the sample covariance matrix $P^e$, which is a low-rank approximation of the true covariance matrix $P$. It is well known by those who use the EnKF in weather forecasting applications that the finite size of the ensemble in the EnKF causes spurious correlations, which

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**Figure 3.** Error norm in the estimation of an $Re_T = 100$ flow using the EnKF, starting from bad initial conditions. The localization used is given in (2.3), with localization length scales of (50,50,25), as motivated by Figure 1. The $- - -$ line represents the statistics of the error before estimation was attempted (averaged over 54 flow-through times), and the $- ◭ -$ line represents the error norm of the EnKF at statistical steady state.

**Figure 4.** As in Figure 3, but reporting the correlation.
in practice must be damped using a distance-dependent localization function in order to ensure adequate convergence of the estimator.

This paper has presented the first near-perfect estimator tracking of an $Re = 100$ turbulent flow using wall information only; that is, we have demonstrated a sustained synchronization of the state estimate with the truth when an accurate (but not perfect) initial condition is used in the estimator, and the localization function is tuned appropriately. It was found that the ability to achieve such tracking is somewhat sensitive to the precise values of the parameters used to define the localization function.

When starting from a bad initial condition, though synchronization was not achieved in our simulations, an order of magnitude less error at 20 viscous units from the wall was achieved than the best existing result in the published literature. Based on the simulations reported in both the tracking and full estimation problems, this reduced level of error in the unsynchronized simulations appears to be rather insensitive to the precise values of the parameters used to define the localization function.

Besides further tuning of the localization function used and the parameters defining it, one interesting possibility for improving the present estimation strategy is to implement a “Rogallo transform” (see, Rogallo 1981 and Rogallo et al. 1984) for the quantities being estimated. In his pioneering work, Rogallo showed that, in regions of high shear (in this case, near a wall), a convenient transformation on the domain may be defined that moves something like a windshield wiper. Such a transformation on the domain might in the present problem provide a significantly slower evolution of the individual discretized flow perturbation quantities being estimated, thus creating a significantly easier problem for the EnKF to analyze. Another promising idea is to explore recent hybrid methods for state estimation that consistently combine the strengths (and numerical tractability) of the EnKF and 4Dvar approaches.

Once the best estimator possible for this problem has been developed, of course, the problem of controlling a turbulent flow based on this estimate must be revisited, as well as the extension of this approach to slightly higher Reynolds numbers. The present investigation, which represents the first reasonably high-fidelity estimate of a turbulent flow based on wall information only, represents a significant step in this direction.

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