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A noncausal framework for model-based feedback control of spatially developing perturbations in boundary-layer flow systems. Part I: formulation

Patricia Cathalifaud*, Thomas R. Bewley

Flow Control Lab, Department of MAE, UC San Diego, La Jolla, CA 92093-0411, USA

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Abstract

We present a noncausal framework for model-based feedback stabilization of a large class of spatially developing boundary-layer flow systems. The systems considered are (approximately) parabolic in the spatial coordinate x. This facilitates the application of a range of established feedback control theories which are based on the solution of differential Riccati equations which march over a finite horizon in x (rather than marching in t, as customary). However, unlike systems which are parabolic in time, there is no causality constraint for the feedback control of systems which are parabolic in space; that is, downstream information may be used to update the controls upstream. Thus, a particular actuator may be used to neutralize the effects of a disturbance which actually enters the system downstream of the actuator location. In the present paper (Part I), a numerically tractable feedback control strategy is formulated which takes advantage of this special capability of feedback control rules in the spatially parabolic setting in order to minimize a globally defined cost function in an effort to maintain laminar boundary-layer flow. A companion paper (Part II) presents numerical simulations which verify the effectiveness of the present approach.

Keywords: Flow control; Noncausal filtering/smoothing; Discrete delta formulation

1. Introduction

This paper considers the feedback estimation and control of small, spatially developing, three-dimensional perturbations to a thin laminar boundary layer in a viscous wall-bounded flow. Control is applied via a blowing/suction distribution over a portion of the wall, and state estimation is accomplished via measurements of skin friction and pressure distributed over the same region. The wall-normal direction is taken to be y and the leading edge of the surface, which might be blunt, is near the line defined by $x = y = 0$; the wall thus lies in the half plane $\{y = 0, x \gtrsim 0\}$. In the special case of an unswept flat plate, the streamwise direction is x and the spanwise direction is z. More generally, the leading edge of the surface over which the boundary layer develops may be swept, and the surface may be inclined and/or curved in the x–y plane. The curvilinear coordinate system is fitted to the body such that the surface is defined by $\{y = 0, x \gtrsim 0\}$ even

* Corresponding author. Tel.: +1-858-5344287; fax: +1-858-8223107.
E-mail addresses: catalifo@turbulence.ucsd.edu (P. Cathalifaud), bewley@ucsd.edu (T.R. Bewley).

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when the leading edge is swept and the surface is curved. Special cases of interest included in the framework presented here include the stabilization of the Blasius, Falkner-Skan, Falkner-Skan-Cooke, and Görtler families of boundary-layer flows.

An important characteristic of laminar systems of this type, which fall under the classic "boundary-layer assumption", [1] is that they are essentially independent of time [2], and the equations that govern them, subject to the correct approximations, are *parabolic in* x. Further generalizations to the framework presented here, such as accounting for heat transfer to or from the surface, are straightforward extensions as long as the boundary-layer assumption remains valid.

Hill [12] pointed out the role of adjoint systems in the local receptivity problem for boundary-layer flow systems. By using an iterative adjoint-based optimization strategy, Luchini [13] and Andersson et al. [1] found the worst-case (a.k.a. "optimal") perturbations of the boundary-layer flow that lead to a maximum energy growth of the perturbations. Iterative (adjoint-based) control optimization strategies for boundary-layer flow systems are appropriate for open-loop control optimizations, and are beginning to see successful applications in this regard. For recent reviews of this line of research, see, e.g., Walter et al. [20], Cathalifaud and Luchini [7], and Pralits et al. [16]. However, it is computationally quite difficult (if not impossible) to apply iterative, adjoint-based control optimization strategies in the closed-loop setting to neutralize the effects of the random flow disturbances that arise in nature. For such problems, feedback control strategies which can respond quickly and in a coordinated fashion to measurements of the flow system are necessary.

There is a large body work in the controls literature on the feedback estimation and control of systems which are parabolic in time. Of particular interest for non-normal systems, such as those often encountered in fluid mechanics, is the fact that $\mathscr{H}_2/\mathscr{H}_\infty$ control theory, which is quite well suited to such systems, is now well understood for both infinite-horizon and finite-horizon control problems, and is discussed in detail in standard textbooks (see, e.g., [9]). Applications of this and related feedback control theories to fluid-mechanical systems generally reduce the non-normality of the system eigenvectors by closing the feedback control loop (see [5]), thereby rendering such systems much better behaved. Though subtle issues related to the infinite dimension and inflow/outflow conditions make the application of established feedback control strategies to such systems nontrivial, significant progress has been made in recent years. For a recent review of this active area of research, see [4]. The present paper develops a closed-loop, Riccati-based *feedback* control strategy (as opposed to an open-loop, adjoint-based control *optimization* strategy) for a spatially developing boundary layer flow system. The present work differs from all previous investigations of the Riccati-based feedback control of fluid systems in that it leverages the *parabolic* evolution of boundary layer flow systems in space in order to reduce the dimension of the Riccati equations to be solved in the formulation of the feedback control equations in order to make them numerically tractable. This provides an attractive alternative to the more common *parallel* flow assumption, also referred to as the assumption of "spatial homogeneity", or "spatial invariance" of the base flow, which facilitates the use of Fourier transforms to decouple the problem of the control of flow perturbations at each wavenumber pair; see [3–5] and [11] for further discussion of this alternative approach.

Control strategies for systems which evolve parabolically in time must be *causal*; that is, they must depend only on present and past measurements of the flow. However, control strategies for systems which evolve parabolically in space are not limited by such a constraint; the control at a particular actuator location may depend on measurements taken both upstream and downstream. Thus, to exploit the additional measurement information available in this setting, a different set of tools is needed for this problem beyond the standard

[1] The boundary layer assumption is that the boundary-layer thickness is much smaller than the streamwise length scales in the system, and that the time scale of the external perturbations to the system are large with respect to the boundary-layer thickness divided by the freestream velocity (see, e.g., [17]).

[2] Time variations in the system model are easily accounted for by gradual variation of the inflow conditions and the external disturbances.

LQG ($\mathscr{H}_2$) framework and "robustifying" extensions thereof ($\mathscr{H}_\infty$, LTR, etc.). Fortunately, many of the necessary control theoretic tools for the present problem were essentially laid out by Anderson and Moore [2] and Middleton and Goodwin [14], though with very different applications in mind. The present paper discusses the several additional considerations necessary to synthesize these tools and apply them to boundary-layer flow systems.

Unlike recent efforts to develop decentralized feedback control strategies for boundary-layer flows, which depend only upon flow measurements and state estimates in the immediate vicinity of any given actuator, the present approach sacrifices localization of the feedback rules in the streamwise coordinate in order to achieve possibly significant performance improvements over that possible with localized strategies. Performance comparisons are conducted in Part II of the present study; the purpose of the present paper (Part I) is simply to present a numerically tractable noncausal framework for the feedback stabilization of boundary-layer flow systems.

2. Governing equations

Based on the dimensional coordinates $\{x^*, y^*, z^*\}$, velocities $\{u^*, v^*, w^*\}$, and pressure p^*, we define the dimensionless quantities $x = x^*/L$, $\{y, z\} = \{y^*, z^*\}/\delta$, $u = u^*/U_0$, $\{v, w\} = \{v^*, w^*\}Re_\delta/U_0$, and $p = p^* Re_\delta^2/(\rho U_0^2)$, where U_0 is the freestream velocity, ρ is the density, μ is the viscosity, $\nu = \mu/\rho$ is the kinematic viscosity, L is a reference streamwise length, $\delta = \sqrt{L\nu/U_0}$ is a reference boundary layer thickness, and $Re_\delta = U_0\delta/\nu$ is a reference Reynolds number. Also, from the dimensional radius of curvature r^* of the surface in the x–y plane, we define the dimensionless curvature parameter $\varepsilon = \delta/|r^*|$, the Görtler number $G = Re_\delta\sqrt{\varepsilon}$, and a sign function s such that $s = 0$ corresponds to a flat wall, $s = 1$ corresponds to a concave wall, and $s = -1$ corresponds to a convex wall.

In order to apply the boundary-layer approximation and to develop a linear set of equations governing small perturbations to the nominal (undisturbed) boundary-layer flow, we make the following assumptions:

A1: $\delta \ll L$ (i.e., $Re_\delta \gg 1$);
A2: $\delta \ll |r^*|$ (i.e., $\varepsilon \ll 1$);
A3: $G \lesssim O(1)$;
A4: the nominal (undisturbed) flow is laminar and steady.

Note that the boundary-layer approximation of the Navier–Stokes equations is not valid in the vicinity of the leading edge. The present work avoids this singularity by considering the evolution of the system only over the interval over which the control is applied, which we define as $x_0 \leqslant x \leqslant L$, where $x_0 > 0$. In order to develop control strategies which are not sensitive to errors in the modeling of the flow upstream of x_0, we will seek control strategies which are insensitive to small errors in the nominal inflow velocity profile.

Though not necessary for the application of the present control approach, it is convenient to approximate the nominal boundary layer flow $\{U(x, y), V(x, y), W(x, y)\}$ by a profile of the Blasius/Falkner–Skan–Cooke/Görtler family (see, e.g., [8]). Similarity solutions of this commonly occurring class of boundary-layer flows may be found by solving the coupled ODEs

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0, \quad g'' + \frac{m+1}{2} f g' = 0,$$

$$f(0) = f'(0) = 0, \quad f'(\infty) \to 1, \quad g(0) = 0, \quad g(\infty) \to 1,$$

by defining $U_0 = x^m$ and $\eta = y\sqrt{U_0/x}$, and taking $U = U_0 f'(\eta)$, $W = W_0 g(\eta)$, and $V = \sqrt{U_0/x}[(1-m)\eta f'(\eta) - (1+m) f(\eta)]/2$. Alternatively, for systems in which, e.g., the curvature of the wall changes gradually as a function of x (as with the flow over a typical airfoil), the nominal boundary-layer flow profile $\{U(x, y), V(x, y),$

$W(x,y)\}$ may be found via straightforward numerical integration of the steady-state boundary-layer equations over the appropriate geometry.

Small three-dimensional perturbations to the nominal flow, $\{u(x,y,z), v(x,y,z), w(x,y,z)\}$, are governed by the linearized Navier–Stokes equation. As the system governing these perturbations is linear and homogeneous in z, we may decouple the various spanwise modes of this system by taking the Fourier transform of all perturbation variables with spanwise variation (namely, the state, the controls, the measurements, and the disturbances) in the z direction (see, e.g., [5]). In the present discussion, we therefore consider a particular Fourier mode of the flow perturbations, and assign a variation in z of $\exp(-\mathrm{i}\beta z)$ to all of these variables. Once the control problem is solved for a series of spanwise wavenumbers, inverse Fourier transform of the feedback gains lead to feedback convolution kernels which are spatially localized in the spanwise coordinate, as shown in Part II of this work. Such localization in the spanwise coordinate of the feedback convolution kernels greatly facilitate their practical implementation (for further discussion see, e.g., [3,4]).

Following the analysis of Hall [10], under the boundary-layer assumptions itemized above, the linearized, nondimensional equations for the flow perturbations reduce to

$$
\begin{aligned}
&(Uu)_x + Vu_y + U_y v - \mathrm{i}\beta W u - u_{yy} + \beta^2 u = 0,\\
&Uv_x + V_x u + (Vv)_y + p_y - \mathrm{i}\beta W v + 2sG^2 U u - v_{yy} + \beta^2 v = 0,\\
&Uw_x + W_x u + V w_y + W_y v - \mathrm{i}\beta p - \mathrm{i}\beta W w - w_{yy} + \beta^2 w = 0,\\
&u_x + v_y - \mathrm{i}\beta w = 0,
\end{aligned} \tag{1}
$$

with the boundary and initial conditions:

$$
\begin{aligned}
&u = w = 0, \quad v = v_w(x) \quad \text{at } y = 0,\\
&u = v = w = 0 \quad \text{at } y = \infty,\\
&\{u, v, w\} = \{u_0, v_0, w_0\} \quad \text{at } x = x_0,
\end{aligned} \tag{2}
$$

where $v_w(x)$ is the control velocity of blowing and suction distributed over the wall on the strip $x_0 < x < L$. The purpose of the control in this problem is to keep the flow perturbations sufficiently small that transition to turbulence is inhibited.

Define the normal vorticity $\eta^* = \partial u^*/\partial z^* - \partial w^*/\partial x^*$ and the corresponding dimensionless form $\eta = -\mathrm{i}\beta u - w_x/Re_\delta^2$. We now combine the governing equations (1) in such a way as to determine a set of two coupled equations for the perturbation components of the normal velocity and normal vorticity. The first of these equations is found by taking the Laplacian of the second component of the momentum equation, substituting the expression for Δp found by taking the divergence of the momentum equation, and applying continuity. The second of these equations is found by taking the normal component of the curl of the momentum equation. Defining $D^k = \partial^k/\partial y^k$, the result is

$$
\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & \tilde{E}_{22} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}, \tag{3}
$$

where $\tilde{E}_{11} = U(D^2-\beta^2) - U_{yy}$, $\tilde{E}_{12} = -(2\mathrm{i}/\beta)[U_{xy} + U_x D^1]$, $\tilde{E}_{22} = -U$, $\tilde{A}_{11} = -[(V_{yy} - \beta^2 V)D^1 + V_{yyy} + V_y(D^2 - \beta^2) + VD^3 - D^4 + 2\beta^2 D^2 - \beta^4 + \mathrm{i}\beta W_{yy} - \mathrm{i}\beta W D^2 + \mathrm{i}\beta^3 W]$, $\tilde{A}_{21} = -\mathrm{i}\beta U_y$, $\tilde{A}_{22} = [U_x + VD^1 - D^2 + \beta^2 - \mathrm{i}\beta W]$, and $\tilde{A}_{12} = -(\mathrm{i}/\beta)[V_{xyy} - V_x(D^2+\beta^2) + 2\mathrm{i}\beta(W_x D^1 - W_{xy}) - 2\beta^2 G^2 U]$.

3. State-space formulation: discretization in y

We now perform a discretization of the system in the y coordinate on a finite number of discretization points with the appropriate grid stretching. Let $\{\mathbf{v}, \boldsymbol{\eta}\}$ denote the spatial discretizations of $\{v, \eta\}$ on the interior of the domain. The derivative operators D^k may be approximated in this discretization using any of a variety of techniques, such as finite differences, Padé, Chebyshev, etc. Define the matrices $\{\hat{E}_{11}, \hat{E}_{12}, \hat{E}_{22}, \hat{A}_{11}, \hat{A}_{12}, \hat{A}_{21}, \hat{A}_{22}\}$ as the spatial discretizations of $\{\tilde{E}_{11}, \tilde{E}_{12}, \tilde{E}_{22}, \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22}\}$ on the interior of the domain using the chosen technique, and the vectors $\mathbf{e}_{11}$ and $\mathbf{a}_{11}$ to denote the influence of the normal velocity at the wall on, respectively, the left-hand side and right-hand side of the v component of the discretization of (3). Using these discrete forms, it is straightforward to express (3) in the state-space form

$$\mathbf{q}_x = A\mathbf{q} + B\phi, \tag{4}$$

$$\mathbf{q} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\eta} \\ v_w \end{pmatrix}, \quad A = \begin{pmatrix} \hat{E}^{-1}\hat{A} & \hat{E}^{-1}\mathbf{a} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\hat{E}^{-1}\mathbf{e} \\ 1 \end{pmatrix},$$

$$\hat{E} = \begin{pmatrix} \hat{E}_{11} & \hat{E}_{12} \\ 0 & \hat{E}_{22} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \mathbf{e}_{11} \\ 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \mathbf{a}_{11} \\ 0 \end{pmatrix}.$$

The control variable in this formulation is $\phi = \mathrm{d}v_w/\mathrm{d}x$.

4. System disturbances and discretization in x

To account for external system disturbances and modeling uncertainties, we now modify the state equation (4) by adding disturbances $\mathbf{w}$ to the right-hand side:

$$\mathbf{q}_x = A\mathbf{q} + B\phi + D\mathbf{w}, \tag{5}$$

where the disturbance vector $\mathbf{w}$ depends on the spatial coordinate x. We desire to develop a global strategy in which the control $\phi(x)$ may actually respond to disturbances $\mathbf{w}(x)$ acting over the entire domain under consideration $x_0 \leqslant x \leqslant L$. To facilitate this in the standard (causal) setting, we first discretize the system in x, then define an augmented state

$$\mathbf{q}_k^a = \begin{pmatrix} \mathbf{q}_k \\ \mathbf{q}_k^{\mathbf{w}} \end{pmatrix} \tag{6}$$

at each station $x_k = x_0 + k\Delta$, $k = 0, \ldots, N$, where $\Delta = (L - x_0)/N$ represents the grid spacing in x, $\mathbf{q}_k = \mathbf{q}(x_k)$, $\mathbf{w}_k = \mathbf{w}(x_k)$, and

$$\mathbf{q}_0^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \end{pmatrix}, \quad \mathbf{q}_1^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_N \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{q}_N^{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_N \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that the augmented state $\mathbf{q}_k^a$ at a particular streamwise station x_k need only include the disturbances entering the system downstream of that location, as the influence of the disturbances upstream are accounted for in $\mathbf{q}_k$. Note also that we can express the evolution of $\mathbf{q}_k^{\mathbf{w}}$ in the discrete state-space form

$$\mathbf{q}_{k+1}^{\mathbf{w}} = A^d \mathbf{q}_k^{\mathbf{w}}, \quad A^d = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}, \tag{7}$$

where the relation between $\mathbf{w}_k$ and $\mathbf{q}_k^{\mathbf{w}}$ is

$$\mathbf{w}_k = M^{\mathbf{w}} \mathbf{q}_k^{\mathbf{w}}, \quad M^{\mathbf{w}} = (\, I \quad 0 \quad \cdots \quad 0\,). \tag{8}$$

By combining Eqs. (5), (7), and (8), we can obtain a state-space formulation for the augmented state $\mathbf{q}^a$. However, the inherently discrete nature of the evolution of our disturbance model $\mathbf{q}^{\mathbf{w}}$ compels us to first derive a discrete formulation of the state equation (5). To accomplish this, we approximate $\{A, B, \mathbf{q}, \phi\}$ with $\{A_k, B_k, \mathbf{q}_k, \phi_k\}$ over the interval $x_k \leqslant x < x_{k+1}$ for each value of k, where, e.g., $A_k = A(x_k)$. Using this approximation (commonly referred to as a "zero-order hold"), we may express (5) in the following "delta form" [14]:

$$\delta \mathbf{q}_k = \Omega_k A_k \mathbf{q}_k + \Omega_k B_k \phi_k + \Omega_k D_k \mathbf{w}_k, \tag{9}$$

where $\Omega_k = (1/\Delta) \int_0^{\Delta} \exp(A_k \tau)\, \mathrm{d}\tau$ and $\delta \mathbf{q}_k = (\mathbf{q}_{k+1} - \mathbf{q}_k)/\Delta$. Note in particular that $\Omega_k \to I$ as $\Delta \to 0$, and thus the discrete-in-x relation (9) tends towards the continuous-in-x relation (5) as the grid is refined. This behavior of the δ-formulation also follows for the Riccati and Lyapunov equations that arise in the control and estimation problems in the following sections, and is an appealing characteristic of this particular discrete formulation. Note that the calculation of the matrix exponential necessary to determine Ω_k can be performed with any of at least 19 "dubious" techniques [15]. One of the least dubious of these techniques, which appears to be adequate for the present system for sufficiently small Δ, is the so-called scaling and squaring method. Combining (9), (7), and (8), we finally obtain a discrete, causal state-space formulation for the augmented state, to which standard control theories may be applied:

$$\delta \mathbf{q}_k^a = A_k^a \mathbf{q}_k^a + B_k^a \phi_k, \tag{10}$$

where

$$A_k^a = \begin{pmatrix} \Omega_k A_k & \Omega_k D_k M^{\mathbf{w}} \\ 0 & A^d \end{pmatrix} \quad \text{and} \quad B_k^a = \begin{pmatrix} \Omega_k B_k \\ 0 \end{pmatrix}.$$

5. Optimal control for noncausal systems

In the original PDE setting, our control objective may be written as finding a feedback control rule which minimizes the cost function

$$\mathscr{J} = \int_{x_0}^{L} \left[\int_0^{\infty} (\alpha_1^2 v^* v + \alpha_2^2 \eta^* \eta)\, \mathrm{d}y + \alpha_3^2 v_w^* v_w + \alpha_4^2 \phi^* \phi \right] \mathrm{d}x.$$

Discretizing in x and y, the cost function may be approximated by

$$\mathscr{J} = \sum_{i=0}^{N} \Delta[(\mathbf{q}^a)_i^* Q^a \mathbf{q}_i^a + \alpha_4^2 \phi_i^* \phi_i], \tag{11}$$

where

$$Q^a = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \alpha_1^2 I_s & 0 & 0 \\ 0 & \alpha_2^2 I_s & 0 \\ 0 & 0 & \alpha_3^2 \end{pmatrix},$$

and I_s is a diagonal matrix with the corresponding local grid spacing on the elements of the diagonal.

Note that the technique of augmenting the initial state with the disturbances entering the entire system in (6) facilitated the conversion of the noncausal problem described in the introduction into the causal problem represented by (10). Together with the control objective (11), a feedback control rule of the form

$$\phi_k = -K_{k+1} \mathbf{q}_k^a \tag{12}$$

may be found directly using standard "discrete-time" optimal control theory. In fact, as discussed in Bitmead et al. [6], the Riccati equation associated with this control problem may be partitioned in a convenient fashion by defining

$$\begin{aligned} K_k &= (\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k)^{-1} B_k^* \Omega_k^* (K_k^1 \quad K_k^2), \\ K_k^1 &= \Sigma_k^{11}(I + \Delta \Omega_k A_k), \quad K_k^2 = \Delta \Sigma_k^{11} \Omega_k D_k M^{\mathbf{w}} + \Sigma_k^{12}(I + \Delta A_k^d), \end{aligned} \tag{13}$$

where Σ_k^{11} and Σ_k^{12} solve the Riccati and Lyapunov equations

$$\begin{aligned} \bar{\delta}\Sigma_k^{11} &= Q + A_k^* \Omega_k^* \Sigma_k^{11} + \Sigma_k^{11} \Omega_k A_k + \Delta A_k^* \Omega_k^* \Sigma_k^{11} \Omega_k A_k \\ &\quad -(K_k^1)^* \Omega_k B_k [\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k]^{-1} B_k^* \Omega_k^* K_k^1, \\ \bar{\delta}\Sigma_k^{12} &= A_k^* \Omega_k^* \Sigma_k^{12} + [I + \Delta A_k^* \Omega_k^*][\Sigma_k^{11} \Omega_k D_k M^{\mathbf{w}} + \Sigma_k^{12} A^d] \\ &\quad - (K_k^1)^* \Omega_k B_k [\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k]^{-1} B_k^* \Omega_k^* K_k^2, \end{aligned} \tag{14}$$

where $\bar{\delta}\Sigma_k = -(\Sigma_k - \Sigma_{k-1})/\Delta$. As $\Delta \to 0$, Eqs. (14) tend towards the corresponding continuous Riccati and Lyapunov equations (cf. [14]).

Finally, by combining (12) and (10), we can express ϕ_k as a simple function of the initial augmented state vector $\mathbf{q}_0^a$:

$$\phi_k = K_{k+1}^0 \mathbf{q}_0^a, \tag{15}$$

where

$$K_{k+1}^0 = -K_{k+1} \prod_{i=0}^{k-1} (A_i^a - B_i^a K_{i+1}).$$

6. Optimal estimation/smoothing

By (15), we see that we can express the optimal control distribution on $x_0 < x < L$ which minimizes the globally defined cost function $\mathscr{J}$ as a simple function of the upstream flow perturbation $\mathbf{q}_0$ and the system disturbances $\mathbf{w}(x)$ between x_0 and L. The task which remains is to find a simple way to obtain a good estimate of $\mathbf{q}_0^a$ based on the available measurements at the wall. Defining the vector $\boldsymbol{\mu}$ as the measurement noise, the measurements of the streamwise and spanwise skin friction and pressure distributions over the wall may be written as

$$\mathbf{y}(x)=\begin{pmatrix}\left.\dfrac{\partial u}{\partial y}\right|_{\text{wall}}(x)\\ \left.\dfrac{\partial w}{\partial y}\right|_{\text{wall}}(x)\\ p|_{\text{wall}}(x)\end{pmatrix}+\boldsymbol{\mu}. \tag{16}$$

Note that applying the nondimensionalization discussed previously to the definition of η, to the continuity equation, and to the wall-normal momentum equation, it is straightforward to write

$$\left.\frac{\partial \eta}{\partial y}\right|_{\text{wall}}=-\mathrm{i}\beta\left.\frac{\partial u}{\partial y}\right|_{\text{wall}}-\frac{1}{Re_\delta^2}\frac{\partial}{\partial x}\left.\frac{\partial w}{\partial y}\right|_{\text{wall}},$$

$$\left.\frac{\partial^2 v}{\partial y^2}\right|_{\text{wall}}=-\frac{1}{2U_w^{(1)}}\left[-\frac{\partial U_w^{(1)}}{\partial x}\;\delta^1/\delta y^1\big|_w+\delta^4/\delta y^4\big|_w-\beta^2\;\delta^2/\delta y^2\big|_w\right]u+\mathrm{i}\beta\left.\frac{\partial w}{\partial y}\right|_{\text{wall}}+\frac{U_3^{(3)}}{2U_w^{(1)}}v_w,$$

$$\left.\frac{\partial^3 v}{\partial y^3}\right|_{\text{wall}}=\left(\beta^2-\frac{1}{Re_\delta^2}\frac{\partial^2}{\partial x^2}\right)p|_{\text{wall}}-U_w^{(1)}\frac{\partial v_w}{\partial x}, \tag{17}$$

where the notation $\delta^k/\delta y^k|_w$ denotes the discretization of the kth derivative operator evaluated at the wall and $U_w^{(k)}$ the kth y-derivative of U evaluated at the wall.

By neglecting the terms in $1/Re_\delta^2$ in (17), we can express the skin friction and pressure at the wall as

$$\begin{pmatrix}\left.\dfrac{\partial u}{\partial y}\right|_{\text{wall}}\\ \left.\dfrac{\partial w}{\partial y}\right|_{\text{wall}}\\ p|_{\text{wall}}\end{pmatrix}=Z\mathbf{q}+N\phi, \tag{18}$$

where

$$Z=\begin{pmatrix}0 & \dfrac{\mathrm{i}}{\beta}\left.\dfrac{\delta^1}{\delta y^1}\right|_w\\ -\dfrac{\mathrm{i}}{\beta}\left.\dfrac{\delta^2}{\delta y^2}\right|_w & \dfrac{1}{2\beta^2U_w^{(1)}}\left[-\dfrac{\partial U_w^{(1)}}{\partial x}\left.\dfrac{\delta^1}{\delta y^1}\right|_w+\left.\dfrac{\delta^4}{\delta y^4}\right|_w-\beta^2\left.\dfrac{\delta^2}{\delta y^2}\right|_w\right]\\ \dfrac{1}{\beta^2}\left.\dfrac{\delta^3}{\delta y^3}\right|_w & 0\end{pmatrix}\tilde{M}+Z_w,$$

$$N = \begin{pmatrix} 0 \\ 0 \\ 1/\beta^2 U_w^{(1)} \end{pmatrix}, \quad Z_w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\mathrm{i}U_w^{(3)}}{2\beta U_w^{(1)}} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{M} = \begin{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & I & 0 \end{pmatrix},$$

where $\tilde{M}\mathbf{q} = \begin{pmatrix} \tilde{\mathbf{v}} \\ \boldsymbol{\eta} \end{pmatrix}$ and $\tilde{\mathbf{v}}$ denotes the y-discretization of the normal velocity that includes the velocity at the wall v_w.

Using relations (16) and (18), we can approximate the vector of the wall measurements $\mathbf{y}$ as a function of the discrete state vector $\mathbf{q}$, the control variable ϕ, and the measurement noise $\boldsymbol{\mu}$:

$$\mathbf{y} = Z\mathbf{q} + N\phi + \boldsymbol{\mu}. \tag{19}$$

Applying the definition of the augmented state $\mathbf{q}^a$, we may write (19) as

$$\mathbf{y}_k = Z^a\mathbf{q}_k^a + N_k\phi_k + \boldsymbol{\mu}_k, \tag{20}$$

where $Z^a = (Z \;\; 0)$. We now define the notation $\hat{\mathbf{q}}_{k|m}^a = \hat{\mathbf{q}}^a(x_k|x_m)$ to denote the estimate of $\mathbf{q}^a(x_k)$ based on the measurements $\mathbf{y}(x)$ from $x_0 \leqslant x \leqslant x_m$. Our aim is to calculate an estimate of $\mathbf{q}_0^a$ based on the measurements $\mathbf{y}(x)$ for $x_0 \leqslant x \leqslant x_N = L$ (i.e. $\hat{\mathbf{q}}_{0|N}^a$). This is a "smoothing" problem, and, given the correct manipulations, can be solved based on the solution of a standard Kalman filter. To solve this problem, we first substitute the value of ϕ_k obtained in (12) into (10) and (20). Defining $F_k = A_k^a - B_k^a K_{k+1}$ and $H_k = Z_k^a - N_k K_{k+1}$, we have

$$\delta\mathbf{q}_k^a = F_k\mathbf{q}_k^a, \quad \mathbf{y}_k = H_k\mathbf{q}_k^a + \boldsymbol{\mu}_k. \tag{21}$$

Defining $\hat{\mathbf{q}}_{0|-1}^a = E(\mathbf{q}_0^a)$, the a priori estimate of $\mathbf{q}_0^a$, and applying Kalman filter theory to the system (21), we obtain the following evolution equation for the estimate $\hat{\mathbf{q}}_{k|k-1}^a$:

$$\begin{aligned} &\delta\hat{\mathbf{q}}_{k|k-1}^a = F_k\hat{\mathbf{q}}_{k|k-1}^a + L_k[\mathbf{y}_k - H_k\mathbf{q}_{k|k-1}^a], \quad k = 0,1,2,\ldots,N, \\ &L_k = (\Delta F_k + I)P_k H_k^*[\Delta H_k P_k H_k^* + C_\mu]^{-1}, \end{aligned} \tag{22}$$

where P_k is solution of the Riccati equation

$$\delta P_k = P_k F_k^* + F_k P_k + \Delta F_k P_k F_k^* - L_k[\Delta H_k P_k H_k^* + C_\mu]L_k^*, \quad k = 0,1,\ldots,N, \tag{23}$$

where P_0 is an estimate of the covariance of the state $\mathbf{q}_k^a$ at $k = 0$ and C_μ is an estimate of the covariance of the noise $\boldsymbol{\mu}$; in practice, P_0 and C_μ are used as design parameters when developing the estimator. Our problem actually differs a bit from the filtering problem (22). In particular, the information we want to reconstruct, $\mathbf{q}_0^a$, must be obtained from measurements taken on $x_0 \leqslant x \leqslant x_N$. In other words, we seek to determine the value of $\hat{\mathbf{q}}_{0|N}^a$, not the value of $\hat{\mathbf{q}}_{N+1|N}^a$ which can be obtained from (22). As in [2], $\hat{\mathbf{q}}_{0|N}^a$ can be easily derived from the filter problem presented above by marching the discrete equation

$$\hat{\mathbf{q}}_{0|k}^a = \hat{\mathbf{q}}_{0|k-1}^a + \Delta R_k H_k^*[\Delta H_k P_k H_k^* + C_\mu]^{-1}\,[\mathbf{y}_k - H_k\hat{\mathbf{q}}_{k|k-1}^a], \quad k = 0,1,2,\ldots,N, \tag{24}$$

where R_k satisfies the Lyapunov equation

$$\delta R_k = R_k(F_k - L_k H_k)^*, \quad k = 0, 1, 2, \ldots, N,$$

where $R_0 = P_0$; note that $\hat{\mathbf{q}}^a_{0|-1} = E(\mathbf{q}^a_0)$ as stated previously.

Assuming that the initial state $\mathbf{q}_0$ is a random variable uncorrelated with the disturbances $\boldsymbol{\mu}_k$, it is straightforward to partition this estimation problem as we did previously with the control problem (see Appendix A).

We thus obtain $\hat{\mathbf{q}}^a_{0|N}$, which is the best approximation possible of the initial augmented state $\mathbf{q}^a_0$ given all of the measured data on $x_0 \leqslant x \leqslant L$. This estimate of the augmented state at x_0 may then be combined with the control relationship (15) to determine the optimal control based on the available noisy measurements.

7. Robustness

In order to maintain effectiveness in the control and estimation problems even in the presence of adversely structured state disturbances and measurement noise, it is important to analyze and possibly supplement the robustness of our present formulation. The state-space formulation (10) of our problem has the peculiar feature that the system disturbances are included *inside* the state vector $\mathbf{q}^a_k$. By applying standard $\mathscr{H}_2$ control theory to the problem (10), we determine the most effective control ϕ_k in response to the state vector $\mathbf{q}^a_k$ (that is, the state $\mathbf{q}_k$ together with the external disturbances between x_k and x_N). In other words, the control strategy responds optimally to any disturbances (including those with adverse structure), as the disturbances themselves are part of the augmented state. Thus, robustness to external disturbances is "built in" to the present state feedback control formulation.

In practice, we do not have any knowledge about these external disturbances, and must estimate them based on the available measurements. Since the disturbances are now included in the state vector $\mathbf{q}^a$, we must solve a standard state estimation problem ($\mathscr{H}_2$ or $\mathscr{H}_\infty$). The solution of the robust estimation problem is an $\mathscr{H}_\infty$ filter that "robustly" estimates the information required to apply the state–feedback control law (12). There are two kinds of uncertainties in this problem: the uncertainty caused by the measurement noise and the uncertainty caused by the unknown initial state, i.e. the uncertainty on the value of $\hat{\mathbf{q}}^a_{0|-1} = E(\mathbf{q}^a_0)$. This $\mathscr{H}_\infty$ filtering problem may be interpreted as a noncooperative game between the estimator, which seeks to find the best estimate of ϕ_k, and nature, which simultaneously seeks to find the most hostile inputs $\boldsymbol{\mu}_k$ (measurements noise) and $\mathbf{q}^a_0$ (initial state). This $\mathscr{H}_\infty$ filtering problem may be express in the following min–max form:

$$\min_{\hat{\mathbf{q}}^a_k} \max_{(\mathbf{q}^a_0, \boldsymbol{\mu}_k)} \mathscr{J} = \sum_{k=0}^{N-1} [(\mathbf{q}^a_k - \hat{\mathbf{q}}^a_{k|k-1})^* K^*_{k+1} Q_k K_{k+1} (\mathbf{q}^a_k - \hat{\mathbf{q}}^a_{k|k-1}) - \gamma^2 (y_k - H_k \mathbf{q}^a_k)^* V_k^{-1} (y_k - H_k \mathbf{q}^a_k)]$$
$$- \gamma^2 (\mathbf{q}^a_0 - \hat{\mathbf{q}}^a_{0|-1})^* P_0^{-1} (\mathbf{q}^a_0 - \hat{\mathbf{q}}^a_{0|-1}),$$

where $\gamma > 0$ represents a specified performance level of the estimator, and Q_k, P_0 and V_k are weighting matrices chosen when developing the estimator. The evolution equation for the estimate $\hat{\mathbf{q}}^a_{k|k-1}$ remains the same as in (22) but with the following filter gain (see e.g. [18,19]):

$$L_k = (\Delta F_k + I) P_k [\Delta H^*_k V_k^{-1} H_k P_k + I - \Delta/\gamma^2 K^*_{k+1} Q_k K_{k+1} P_k]^{-1} H^*_k V_k^{-1}, \tag{25}$$

where

$$\delta P_k = P_k F_k^* + F_k P_k + \Delta F_k P_k F_k^* + (\Delta F_k + I) P_k \,(1/\gamma^2 K_{k+1}^* Q_k K_{k+1} - H_k^* V_k^{-1} H_k) \\ \times [\Delta P_k H_k^* V_k^{-1} H_k + I - \Delta/\gamma^2 P_k K_{k+1}^* Q_k K_{k+1}]^{-1} P_k (\Delta F_k + I)^*, \tag{26}$$

where the initial condition of the Riccati equation is P_0. The weighting matrix P_0 quantifies the uncertainty in the initial conditions $\mathbf{q}_0^a$. In the $\mathscr{H}_\infty$ setting, the estimate $\hat{\mathbf{q}}_{k|k-1}^a$ of $\mathbf{q}_k^a$ has the interesting property that $\hat{\mathbf{q}}_{k|k-1}^a$ depends on the control gain K_{k+1}. This implies a one-way coupling between the control and estimation problems, and it is necessary to solve the control problem first. Note that this coupling is not apparent in the Kalman filter (23), in which a "separation principle" applies.

As we did previously, we may again derive the solution of the smoothing problem from the associated filtering problem. We obtain the following evolution equation for $\hat{\mathbf{q}}_{0|k}^a$:

$$\hat{\mathbf{q}}_{0|k}^a = \hat{\mathbf{q}}_{0|k-1}^a + R_k [\Delta H_k^* V_k^{-1} H_k P_k + I - \Delta/\gamma^2 K_{k+1}^* Q_k K_{k+1} P_k]^{-1} \\ \times H_k^* V_k^{-1} [\mathbf{y}_k - H_k \hat{\mathbf{q}}_{k|k-1}^a], \quad \text{for } k = 0, 1, 2, \ldots, N, \tag{27}$$

where

$$\delta R_k = R_k F_k^* + R_k (1/\gamma^2 K_{k+1}^* Q_k K_{k+1} - H_k^* V_k^{-1} H_k)\,[\Delta P_k H_k^* V_k^{-1} H_k \\ + I - \Delta/\gamma^2 P_k K_{k+1}^* Q_k\, K_{k+1}]^{-1} P_k (\Delta F_k + I)^*, \tag{28}$$

$R_0 = P_0$, and $\hat{\mathbf{q}}_{0|-1}^a$ is the initial condition of (27).

8. Conclusions

The primary challenge in the application of Riccati-based feedback control strategies to fluid-mechanical systems is the enormous state dimension which is necessary to capture such systems with an adequate degree of fidelity. The state dimension necessary to resolve such systems typically renders Riccati-based control strategies numerically unfeasible, and open-loop model reduction strategies are highly prone to misrepresentation of the relevant dynamics of the fluid system, effectively "losing the baby with the bathwater".

In flow systems with two directions of spatial homogeneity (such as channel flows), the linearized system model may be made approachable with Riccati-based feedback control strategies by decoupling the various streamwise and spanwise modes of the problem using Fourier-based approaches. Linearized boundary-layer systems, however, have only one direction of spatial homogeneity.

The present paper proposes a new, Riccati-based feedback control strategy which leverages the fact that linearized boundary-layer systems develop parabolically in the streamwise coordinate. Taking advantage of this property, numerically tractable control and estimation algorithms have been proposed which target the reduction of a globally defined cost function with control feedback, while only requiring the solution of Riccati equations related to system models which are spatially-discretized in a single coordinate direction (y). The state-feedback control strategy used has robustness "built in", as it depends explicitly on the disturbances, which are augmented to the state in the present formulation. The robustification of the estimator via noncooperative analysis is straightforward, and a solution of the "robust" estimation problem which solves this noncooperative game has been presented. The resulting Riccati equations are computationally tractable; numerical results of this formulation which verify its effectiveness are presented in Part II of this work.

Appendix A. Partition of the estimation problem

The Riccati Equation associated with the estimation problem (22) may be partitioned by defining

$$L_k^{(1)} = [[I + \Delta(\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)})]P_k^{11}(Z_k - N_k K_{k+1}^{(1)})^* - [I + \Delta(\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)})]P_k^{12}(N_k K_{k+1}^{(2)})^*$$
$$+ \Delta(\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)})P_k^{12}(Z_k - N_k K_{k+1}^{(1)})^* - \Delta(\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)})P_k^{22}(N_k K_{k+1}^{(2)})^*]\xi_k^{-1},$$

$$L_k^{(2)} = [(I + \Delta A^d)P_k^{12}(Z_k - N_k K_{k+1}^{(1)})^* - (I + \Delta A^d)P_k^{22}(N_k K_{k+1}^{(2)})^*]\xi_k^{-1},$$

where P_k^{11} and P_k^{22} solve the following two Riccati equations:

$$\begin{aligned}\delta P_k^{11} = &P_k^{11}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^* + P_k^{22}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]^*\\
&+[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]P_k^{11} + [\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]P_k^{12}\\
&+ \Delta[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]\, P_k^{11}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^*\\
&+ \Delta[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]\, P_k^{22}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(1)}]^*\\
&+ \Delta[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]\, P_k^{12}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^*\\
&+ \Delta[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]\, P_k^{22}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(1)}]^*\\
&- L_k^{(1)}\xi_k L_k^{(1)^*},\end{aligned}$$

$$\delta P_k^{22} = P_k^{22} A^{d^*} + A^d P_k^{22} + \Delta A^d P_k^{22} A^{d^*} - L_k^{(2)}\xi_k L_k^{(2)^*},$$

and P_k^{12} solves the following Lyapunov equation:

$$\delta P_k^{12} = P_k^{12}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^* + P_k^{22}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]^*$$
$$+ A^d P_k^{12} + \Delta A^d P_k^{12}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^* + \Delta A^d P_k^{22}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]^* - L_k^{(2)}\xi_k L_k^{(1)^*},$$

where

$$\xi_k = \Delta(Z_k - N_k K_{k+1}^{(1)})P_k^{11}(Z_k - N_k K_{k+1}^{(1)})^* - \Delta N_k K_{k+1}^{(2)} P_k^{12}(Z_k - N_k K_{k+1}^{(1)})^*$$
$$- \Delta(Z_k - N_k K_{k+1}^{(1)})P_k^{12}(N_k K_{k+1}^{(2)})^* + \Delta N_k K_{k+1}^{(2)} P_k^{22}(N_k K_{k+1}^{(2)})^* + C_\mu,$$

and $K_k^{(1)}$ and $K_k^{(2)}$ are defined using the relations (13) such that

$$K_k^{(1)} = (\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k)^{-1} B_k^* \Omega_k^* K_k^1, \quad K_k^{(2)} = (\alpha_4^2 I + \Delta B_k^* \Omega_k^* \Sigma_k^{11} \Omega_k B_k)^{-1} B_k^* \Omega_k^* K_k^2.$$

The partition of the smoothing problem (24) involves the solution of the two Lyapunov equations:

$$\delta R_k^{(1)} = R_k^{(1)}[\Omega_k A_k - \Omega_k B_k K_{k+1}^{(1)}]^* + R_k^{(2)}[\Omega_k D_k M^w - \Omega_k B_k K_{k+1}^{(2)}]^*$$
$$- R_k^{(1)}(Z_k - N_k K_{k+1}^{(1)})^* L_k^{(1)^*} + R_k^{(2)} N_k K_{k+1}^{(2)} L_k^{(1)^*},$$

$$\delta R_k^{(2)} = -R_k^{(1)}(Z_k - N_k K_{k+1}^{(1)})^* L_k^{(2)^*} + R_k^{(2)} N_k K_{k+1}^{(2)} L_k^{(2)^*} + R_k^{(2)} A^{d^*}.$$

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