A Fundamental Limit on the Heat Flux in the Control of Incompressible Channel Flow

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***Abstract*—This paper proves that there are no zero-net wall-transpiration control strategies that can sustain net heat flux below the laminar level in an incompressible channel flow with constant-temperature walls. The result represents a fundamental limit on the performance of a controlled nonlinear system as measured by a linear cost function over a broad class of admissible initial conditions and control inputs, not a zero-sum tradeoff in the frequency domain or time domain. Both buoyancy effects (via the Boussinesq approximation) and viscous heating effects are accounted for, and phenomenological justification for the result is also given. The boundedness of solutions of the two-way coupled Navier–Stokes/energy equations (when both buoyancy and viscous heating are accounted for) is also discussed, and a new proof of existence under an appropriate small-data assumption is provided.**

***Index Terms*—Flow control, fundamental performance limits.**

I. Introduction

THE field of flow control has experienced rapid growth in the last decade; see [1], [13], [16], [8], and [17] for recent reviews. Such studies have led to a desire to improve our understanding of canonical flow control problems by quantifying mathematically the fluid-mechanical intuition that the laminar state might represent some sort of *fundamental performance limit* in certain precisely defined flow control problems. Such fundamental limits quantify the best performance possible in a particular system using *any* controls of a given well-defined class. Note that there is a broad body of literature on certain types of "fundamental performance limitations" in the control of systems of ordinary differential equations (ODEs) (see, e.g., Freudenberg and Looze [12], Seron *et al.* [20], and the recent special issue of the IEEE Transactions on Automatic Control dedicated to this topic). However, there are currently, as far as we know, no such fundamental limits rigorously established specifically for the control of Navier–Stokes systems. This paper develops the first result of this sort.

Note that most results referred to as "fundamental performance limitations" in the existing controls literature (the most well known of which being the Bode sensitivity integral) are of "zero-sum" type (i.e., an equality), indicating that an improvement in performance over some interval (either in the frequency domain or in time domain) must necessarily be offset by a degradation in performance over some other interval(s). This result is of a somewhat different nature, as it represents what might be referred to as an "optimality" result for the present (nonlinear) system, establishing a sharp lower bound on the achievable value of a (linear) cost function over any initial conditions of the system and all control distributions over a broad admissible class.

A. System Description: Input, Output, State, and Cost Function

The result proved in this work is of the following type: for a particular nonlinear partial differential equation (PDE) system[1] of the form[2]

$$E\dot{u} = f(u) + g(u)\phi$$
$$z = Cu$$

for any (sufficiently smooth) initial conditions $u(0)$ and all (sufficiently smooth and uniformly bounded) control inputs $\phi(t)$, we establish that

$$J = \lim_{t\to\infty} \frac{1}{t}\int_0^t z(t')dt' \geq J_{\text{laminar}}$$

where the value J_{laminar} is achievable by any control inputs that return the system to a particular state (referred to as the laminar flow). Note that J is linear in the system output z (which itself is linear in the state u); J is not a norm of z. The required regularity on the control distribution ϕ that we will specify below is necessary only for technical reasons in the proof of the proposition establishing that, effectively, the problem we are considering makes sense (that is, that the appropriate norms of the state are, in fact, bounded). The requisite bounds on the control can be set as arbitrarily large; the bounds we establish on the minimum value of the cost function J_{laminar} are independent of these bounds on the control. Also, this paper considers the problem of the direct calculation of controls of the specified

[1]The present system is governed by the Navier–Stokes equation. The control actually enters the system considered as a boundary condition in the present work, not as a right-hand side (RHS) forcing term inside the physical domain. However, using a so-called lifting function (see [15]), it is straightforward to write an equivalent problem with interior forcing, if desired.

[2]Written in this form, $\{E, f(u), g(u), C\}$ are operators, with the state u (below, $\{\mathbf{u}, p, T\}$) defined over a two- or three-dimensional domain Ω, and the control ϕ (below, $\{\phi_+, \phi_-\}$) defined over the upper and lower boundaries of this domain. Note that E is singular (due to the algebraic constraints implied by the continuity equation); thus, this system is sometimes referred to as a differential algebraic equation. Note also that $f(u)$ incorporates an energy-conserving quadratic nonlinearity, and that C is an unbounded operator. The system of interest in this paper is made explicit in (1)–(3) and the paragraphs that follow.

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regularity (e.g., with a model predictive control (MPC) type calculation) and does not consider explicitly the problem of feedback. If a feedback rule were to be used instead, it would have the form of a convolution operator[3] in this PDE formulation. In such a formulation, instead of a bound on the regularity of the control distribution ϕ itself, a bound on the regularity of this convolution operator would be required. That is, an appropriate degree of filtering would be built into this convolution operator in order to insure the requisite smoothness on ϕ, as specified below. An analysis of the smoothness required on the convolution operator in such a feedback setting is deferred to future work, and may be performed with similar energy estimates as considered in this paper.

In this paper, we denote the system input (above, ϕ) as $\{\phi_+, \phi_-\}$ (that is, the blowing/suction distribution on the upper and lower boundaries of the domain), the system state (above, u) as $\{\mathbf{u}, p, T\}$ (that is, the velocity, pressure, and temperature distribution inside the domain Ω, respectively), the system output (above, z) as the scalar $\bar{\Psi}$ (which measures the instantaneous heat flux across the domain Ω), and the cost function measuring the system (above, J) as $\langle\bar{\Psi}\rangle_\infty$ (which measures the infinite-time-averaged heat flux across the domain).

II. Problem Setting

This paper considers an incompressible flow with velocity $\mathbf{u}$, pressure p, and temperature T in a rectangular two-dimensional (2-D) or three-dimensional (3-D) channel Ω governed by the coupled nonlinear PDEs

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p - \mathbf{i}P_x + \mathbf{G}(T) \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = k\Delta T + H(\mathbf{u}). \tag{3}$$

The boundary conditions on the velocity are $\mathbf{u} = -\phi_\pm\,\mathbf{n}$ on $\Gamma_\pm$, where $\phi = \{\phi_+, \phi_-\}$ is the control distribution (to be determined), $\mathbf{n}$ is an outward facing normal vector, Γ_+ denotes the upper wall, and Γ_- denotes the lower wall. The boundary conditions on the temperature are $T|_{\Gamma_\pm} \triangleq T_\pm$, where both T_+ and T_- are assumed to be constant in $x, z,$ and t with $T_+ \neq T_-$. Periodic boundary conditions are applied on $\{\mathbf{u}, p, T\}$ in the x and z directions. The streamwise pressure gradient P_x is specified[4] (e.g., P_x =constant). The parameters $\nu > 0$ and $k > 0$ are assumed to be constant. Without loss of generality, we will shift and scale the temperatures and lengths in the problem such that $T_+ = 1$, $T_- = -1$, and the walls are located at $y = \pm 1$.

The initial conditions $T(t=0) = T_0$ are in $L^2(\Omega)$, and the initial conditions $\mathbf{u}(t=0) = \mathbf{u}_0$ are in $L^2(\Omega)$ and are divergence free, but are otherwise arbitrary.

Define the heat flux through the upper and lower walls at any instant as $\Psi_+ = \int_{\Gamma_+} k(\partial T)/(\partial y)\,d\mathbf{x}$ and $\Psi_- = \int_{\Gamma_-} k(\partial T)/(\partial y)\,d\mathbf{x}$, respectively, and the time average as $\langle\cdot\rangle_t \triangleq (1/t)\int_0^t [\cdot]\,dt'$. Define also the "net heat flux" as $\bar{\Psi} \triangleq (1/2)(\Psi_+ + \Psi_-)$, the "time-averaged net heat flux" as $\langle\bar{\Psi}\rangle_t$, and the "sustained net heat flux" as $\langle\bar{\Psi}\rangle_\infty$. The problem considered in this paper is to find a strict bound on the system output $\langle\bar{\Psi}\rangle_\infty$ for any control inputs ϕ in the broad class[5] $\phi_\pm \in C([0,\infty); H^{1/2}(\partial\Omega)) \cap L^\infty([0,\infty); H^{1/2}(\partial\Omega))$ and $\partial\phi_\pm/\partial t \in C([0,\infty); H^{-1/2}(\partial\Omega)) \cap L^\infty([0,\infty); H^{-1/2}(\partial\Omega))$ satisfying the zero-net mass flux constraint

$$\int_{\Gamma_-} \phi_-\,d\mathbf{x} = \int_{\Gamma_+} \phi_+\,d\mathbf{x} = 0 \quad \forall t > 0. \tag{4}$$

The forcing $\mathbf{G}(T)$ on the momentum equation (2) represents the Boussinesq approximation of the buoyancy force that arises due to temperature fluctuations in the flow. The forcing $H(\mathbf{u})$ on the energy (3) represents the effect of viscous heating. In the following three sections, we consider three cases of interest.

- Case A) This case takes $\mathbf{G}(T) = 0$ and $H(\mathbf{u}) = 0$ (Section III).
- Case B) This case takes $\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = 0$ (Section IV).
- Case C) This case takes $\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla\mathbf{u})$ (Section V).

We then conclude with a discussion in Section VI. The orientation of the gravity vectory $\mathbf{g}$ is arbitrary, so this paper includes both the stable case with $\mathbf{g}$ oriented in the $-y$ direction and the unstable case with $\mathbf{g}$ oriented in the $+y$ direction.

For each of the three cases studied, a proposition is first stated and proved to establish boundedness of $\int_\Omega T^2\,d\mathbf{x}$; then a theorem is stated and proved to establish the fundamental performance limitation. To facilitate readability, these main propositions and theorems are lettered according to the cases to which they correspond. Other minor remarks and corollaries are numbered for later reference.

Remark 1: In the cases with $H(\mathbf{u}) = 0$, for $\mathbf{u}_0 \in L^2(\Omega)$ with $\nabla \cdot \mathbf{u} = 0$ and $T_0 \in L^2(\Omega)$, the existence for all $t > 0$ of weak solutions to (1)–(3) with boundary control applied (with the smoothness prescribed above) is well known[6] [22]. (In the case with $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla\mathbf{u})$, we will present a formal proof of existence under an appropriate small data assumption in Section V, Remark 5.) However, the uniqueness of these solutions is known only when the domain is 2-D (see [22]). In this paper, we will work only with weak solutions of (1)–(3), and thus the rigorous proofs presented below are valid in both 2-D and 3-D.

We will have occasion to leverage several fundamental inequalities in this paper, which for convenience are reviewed briefly here. Following [21] (see, e.g., [3, Section 1.5] for a succinct summary), with $n = 2$ or 3, we take

$$H = \{\mathbf{u} \in (L^2(\Omega))^n;\ \text{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ u_2 = 0 \text{ on } \Gamma_+ \cup \Gamma_-,\ \mathbf{u} \text{ periodic in other directions}\}$$

[3]For example, in a closely related problem in [14], a convolution of the form $\phi(x) = \int_\Omega K(x - x')\,u(x')\,dx'$ was used.

[4]It is straightforward to extend this derivation to the case in which P_x is tuned as the flow evolves to maintain a specified (e.g., constant) mass flux in the flow; e.g., in Section IV, the only change necessary is establishing the appropriate bounds on P_x^2 in (15).

[5]Note that H^1 is the classical Sobolev space of functions that are square integrable along with their first derivatives and $H^{1/2}$ is the space obtained by interpolation of L^2 and H^1. Note that the boundary value of functions in $H^1(\Omega)$ are necessarily in $H^{1/2}(\partial\Omega)$. We will also make use of $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in H^1.

[6]Note that weak solutions are, roughly speaking, solutions with finite energy and finite dissipation rate.

and

$$V = \{\mathbf{u} \in (H^1(\Omega))^n;\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} = 0 \text{ on } \Gamma_+ \cup \Gamma_-,\ \mathbf{u} \text{ periodic in other directions}\}$$

and recall the Leray–Hopf projector $\mathcal{P} : (L^2(\Omega))^n \mapsto H$. Then the Stokes operator A is defined as

$$A\mathbf{u} = -\mathcal{P}(\Delta \mathbf{u}) \quad \forall\, \mathbf{u} \in D(A)$$

with $D(A) = V \cap (H^2(\Omega))^n$ and the trilinear form b is defined as

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\, d\mathbf{x} \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

Taking $p \geq 1$ and defining the norms

$$|f|_{L^p} = \left(\int_\Omega |f|^p d\mathbf{x}\right)^{\frac{1}{p}}, \quad \|\mathbf{u}\|_{H^1}^2 = |\mathbf{u}|_{L^2}^2 + |\nabla \mathbf{u}|_{L^2}^2,$$

$$\|\mathbf{u}\|_{H^2}^2 = \|\mathbf{u}\|_{H^1}^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}\right|_{L^2}^2$$

we note the following (see, e.g., [11] and [21]).

Cauchy–Schwarz Inequality: For any a and b defined in Ω

$$\int_\Omega |a|\,|b|\,d\mathbf{x} \leq |a|_{L^2}\,|b|_{L^2}.$$

Poincaré's Inequality: If $\mathbf{u} = 0$ on Γ_+ or Γ_-, there is a numerical constant $\lambda_1 = \lambda_1(\Omega)$ (for the present geometry, $\lambda_1 = \sqrt{2}$) such that

$$|\mathbf{u}|_{L^2} \leq \lambda_1 |\nabla \mathbf{u}|_{L^2} \quad \text{and } |\nabla \mathbf{u}|_{L^2} \leq \lambda_1 |A\mathbf{u}|_{L^2}.$$

Hölder's Inequality:

$$\int_\Omega f_1 \ldots f_n\, d\mathbf{x} \leq |f_1|_{L^{p_1}} \ldots |f_n|_{L^{p_n}}$$

where

$$\frac{1}{p_1} + \cdots \frac{1}{p_n} = 1.$$

Young's Inequality:

$$ab \leq \epsilon \frac{a^p}{p} + \epsilon^{-\frac{q}{p}} \frac{b^q}{q} \quad \text{for } a, b, \epsilon \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Agmon Inequality (in Both 2-D and 3-D):

$$|\mathbf{u}|_{L^\infty} \leq C |\nabla \mathbf{u}|_{L^2}^{1/2} \quad |A\mathbf{u}|_{L^2}^{1/2} \quad \forall\, \mathbf{u} \in D(A).$$

Sobolev Inequalities (in both 2-D and 3-D): For $\mathbf{u}, \mathbf{v}, \mathbf{w}$ sufficiently regular that the corresponding norms are bounded, we have

$$\begin{cases} |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C |\nabla \mathbf{u}|_{L^2}\, |\nabla \mathbf{v}|_{L^2}^{1/2}\, |A\mathbf{v}|_{L^2}^{1/2}\, |\mathbf{w}|_{L^2} \\ |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C |\mathbf{u}|_{L^2}^{1/4}\, |A\mathbf{u}|_{L^2}^{3/4}\, |\nabla \mathbf{v}|_{L^2}\, |\mathbf{w}|_{L^2} \\ |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C |\mathbf{u}|_{L^2}^{1/4}\, |\nabla \mathbf{u}|_{L^2}^{3/4}\, |\nabla \mathbf{v}|_{L^2}\, |\mathbf{w}|_{L^2}^{1/4}\, |\nabla \mathbf{w}|_{L^2}^{3/4} \end{cases}.$$

Gronwall's Lemma: If $dy/dt \leq gy + h$ with g and h integrable, then

$$y(t) \leq y(0) \exp\left(\int_0^t g(\tau)\, d\tau\right) + \int_0^t h(s) \exp\left(\int_s^t g(\tau)\, d\tau\right) ds.$$

We also have (see, for instance, [11] and [21])

$$C_1 |\Delta \mathbf{u}|_{L^2}^2 \leq |A\mathbf{u}|_{L^2}^2 \leq C_2 |\Delta \mathbf{u}|_{L^2}^2$$

$$\sum_{i,j=1}^n \left|\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}\right|_{L^2}^2 \leq C |A\mathbf{u}|_{L^2}^2.$$

Note that the symbols C and C_i are used in this paper to denote positive numerical coefficients that depend on the domain Ω and whose values may be different in each inequality.

III. Case A: Standard Navier–Stokes With Temperature Acting as a Passive Scalar

This section considers the simplest case with $\mathbf{G}(T) = 0$ and $H(\mathbf{u}) = 0$. With this model, the velocity obeys the standard incompressible Navier–Stokes [(1) and (2)] and the temperature acts like a passive scalar obeying a simple unforced convection-diffusion [(3)]. With this model, the velocity affects the temperature field, but the temperature does not affect the velocity field.

Proposition A: In the case with $\mathbf{G}(T) = 0$ and $H(\mathbf{u}) = 0$, the quantities $\int_\Omega T^2\, d\mathbf{x}$ and $|\int_\Omega T\, d\mathbf{x}|$ are uniformly bounded in t.

Proof: Decomposing $T = \bar{T} + T'$, where $\bar{T} = y$, then multiplying (3) by T' and integrating over the domain Ω gives

$$\int_\Omega T' \left[\frac{\partial T'}{\partial t} + \mathbf{u} \cdot \nabla \bar{T} + \mathbf{u} \cdot \nabla T' = k \Delta T'\right] d\mathbf{x}. \tag{5}$$

Integrating the third and fourth terms of (5) by parts, noting (1) and $T'|_{y=\pm 1} = 0$, then applying the Cauchy–Schwarz and Poincaré inequalities gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|T'|_{L^2}^2 + k|\nabla T'|_{L^2}^2 &= -\int_\Omega T'\, \mathbf{u} \cdot \nabla \bar{T}\, d\mathbf{x} \\ &\leq \int_\Omega |T'|\,|\mathbf{u}|\, d\mathbf{x} \\ &\leq |T'|_{L^2}\, |\mathbf{u}|_{L^2} \\ &\leq \lambda_1 |\nabla T'|_{L^2}\, |\mathbf{u}|_{L^2}. \end{aligned}$$

Applying Young's inequality leads to

$$\frac{1}{2}\frac{d}{dt}|T'|_{L^2}^2 + \frac{k}{2}|\nabla T'|_{L^2}^2 \leq \frac{\lambda_1^2}{2k}|\mathbf{u}|_{L^2}^2. \tag{6}$$

By the global existence of weak solutions to (1) and (2), $|\mathbf{u}|_{L^2}$ is uniformly bounded (that is, $|\mathbf{u}|_{L^2} \leq R$) as long as ϕ is uniformly bounded ([15], [21]); thus, applying Poincaré's inequality to the second term

$$\frac{1}{2}\frac{d}{dt}|T'|_{L^2}^2 + \frac{k}{2\lambda_1^2}|T'|_{L^2}^2 \leq \frac{\lambda_1^2}{2k} R^2.$$

Finally, multiplying by the integrating factor $2e^{k\,t'/\lambda_1^2}$ and integrating over $(0,t)$, thereby applying Gronwall's lemma, gives

$$\int_0^t \frac{d}{dt'}\left(e^{k\,t'/\lambda_1^2}|T'|_{L^2}^2\right)dt' \le \int_0^t \frac{\lambda_1^2}{k}R^2\,e^{k\,t'/\lambda_1^2}\,dt'$$
$$|T'(t)|_{L^2}^2 \le |T'(0)|_{L^2}^2 e^{-k\,t/\lambda_1^2} + \frac{\lambda_1^4}{k^2}R^2.$$

Thus, $|T|_{L^2}$ is bounded for all $t>0$. Boundedness of $\int_\Omega |T|\,d\mathbf{x}$, and thus of $|\int_\Omega T\,d\mathbf{x}|$, follows by Cauchy–Schwarz. □

Theorem A: The lowest sustainable net heat flux of an incompressible 2-D or 3-D channel flow [governed by (1)–(3)] with both buoyancy and viscous heating effects neglected [$\mathbf{G}(T)=0$ and $H(\mathbf{u})=0$], when controlled via a distribution of zero-net mass-flux blowing/suction over the channel walls (each with a constant temperature in x, z, and t) is exactly that of the laminar flow.

Proof: Multiplying (3) by T and integrating over the domain Ω gives

$$\int_\Omega T\left[\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = k\Delta T\right]d\mathbf{x}.$$

Integrating the second and third terms by parts, noting (1) and (4) and that $T_+ = 1$ and $T_- = -1$, leads to

$$\frac{1}{2}\frac{d}{dt}|T|_{L^2}^2 + k|\nabla T|_{L^2}^2 = \underbrace{\int_{\Gamma_+} k\frac{\partial T}{\partial y}\,d\mathbf{x} + \int_{\Gamma_-} k\frac{\partial T}{\partial y}\,d\mathbf{x}}_{\Psi_+ + \Psi_-}.$$

Taking the time average of the above expression in the limit that $t\to\infty$, noting that $|T|_{L^2}^2$ is bounded (see Proposition A), leads to the following expression for the sustained net heat flux:

$$\langle\bar{\Psi}\rangle_\infty = \frac{1}{2}\left\langle k|\nabla T|_{L^2}^2\right\rangle_\infty.$$

Again decompose $T = \bar{T} + T'$, where the background profile $\bar{T} = y$. Noting that $T'|_{y=\pm 1} = 0$, it is easily seen that

$$|\nabla T|_{L^2}^2 = \int_\Omega\left[\left(\frac{\partial T'}{\partial x}\right)^2 + \left(\frac{\partial(\bar{T}+T')}{\partial y}\right)^2 + \left(\frac{\partial T'}{\partial z}\right)^2\right]d\mathbf{x}$$
$$= \int_\Omega\left[\Bigg(\underbrace{\frac{\partial \bar{T}}{\partial y}}_{=1}\Bigg)^2 + 2\underbrace{\frac{\partial \bar{T}}{\partial y}}_{=1}\frac{\partial T'}{\partial y}\right]d\mathbf{x} + |\nabla T'|_{L^2}^2$$
$$= 2L_xL_z + |\nabla T'|_{L^2}^2$$

where L_x and L_z are the streamwise and spanwise extent of the domain Ω. Thus

$$\langle\bar{\Psi}\rangle_\infty = kL_xL_z + \frac{k}{2}\left\langle|\nabla T'|_{L^2}^2\right\rangle_\infty.$$

Note that the second term on the right-hand side is nonnegative. The unique steady-state temperature profile minimizing the sustained net heat flux $\langle\bar{\Psi}\rangle_\infty$ is thus given by $T'=0$, that is, by the laminar temperature profile $T=y$. □

Note that unsteady flows (with arbitrary initial conditions) can also achieve this minimizing value of $\langle\bar{\Psi}\rangle_\infty$ (an infinite-time average) simply by exponentially stabilizing the flow state with the laminar temperature profile.

Corollary 1: For the case of a constant pressure-gradient or constant mass-flux channel flow, the unique steady flow corresponding to the laminar temperature profile $T=y$, which minimizes the sustained net heat flux $\langle\bar{\Psi}\rangle_\infty$ as shown in Theorem 1, is given by $\phi = 0$ and the laminar flow profile $\mathbf{u} = C(1-y^2)\mathbf{i}$.

Proof: Denoting the three components of $\mathbf{u}$ as $\{u,v,w\}$, assume that there exists an $\mathbf{x}_0\in\Omega$ and a $t_0>0$ such that $v(\mathbf{x}_0,t_0)\neq 0$. Then, by the parabolic smoothing of (1) and (2), there exists a neighborhood $\mathcal{V}$ of $(\mathbf{x}_0,t_0)$ such that $v(\mathbf{x},t)\neq 0\in\mathcal{V}$. By Theorem A, $T=y$, and therefore $\mathbf{u}\cdot\nabla T = v\neq 0\in\mathcal{V}$. Hence, from (3), $\partial T/\partial t - k\Delta T\neq 0\in\mathcal{V}$, which leads to a contradiction, since $T=y$. Thus, $v=0$ for all $t>0$ and $\mathbf{x}\in\Omega$, and thus all steady-state solutions satisfy $v=0$ everywhere in Ω and $\phi=0$ on the walls. The continuity (1) thus reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

and the momentum (2) reduces, for a steady flow, to

$$-\nu\Delta u + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = -P_x \tag{8a}$$
$$\frac{\partial p}{\partial y} = 0 \tag{8b}$$
$$-\nu\Delta w + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0. \tag{8c}$$

Defining $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ and $\nabla_2 = (\partial/\partial x)\mathbf{i} + (\partial/\partial z)\mathbf{k}$, adding $-\Delta_2 u$ times (8a) to $-\Delta_2 w$ times (8c) and integrating with respect to x and z, then applying integration by parts, periodicity of u, w, and p in x and z, and the continuity (7) gives

$$\int\!\!\int\left[|\Delta_2 u|^2 + |\Delta_2 w|^2 + \left|\frac{\partial}{\partial y}\nabla_2 u\right|^2 + \left|\frac{\partial}{\partial y}\nabla_2 w\right|^2\right]dx\,dz = 0.$$

Thus, $\Delta_2 u = \Delta_2 w = 0$ in Ω, and by periodicity u and w depend only on y. Hence, (8) reduces to

$$-\nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial p}{\partial x} = -P_x \tag{9a}$$
$$\frac{\partial p}{\partial y} = 0 \tag{9b}$$
$$-\nu\frac{\partial^2 w}{\partial y^2} + \frac{\partial p}{\partial z} = 0. \tag{9c}$$

Taking the derivative of (9a) with respect to x and (9c) with respect to z, and adding, we get $\Delta_2 p = 0$ in Ω. By periodicity of p and (9b), we get $p =$ constant in Ω, and therefore (9c) reduces to $(\partial^2 w)/(\partial y^2) = 0$ in Ω. Since $w=0$ on $\Gamma_\pm$, $w=0$ everywhere in Ω. Similarly, (9c) reduces to $(\partial^2 u)/(\partial y^2) = -P_x$ in Ω; since $u=0$ on $\Gamma_\pm$, we find that $u_1 = (P_x/2)(1-y^2)$. □

Remark 2: If $|T_0|$ is bounded, T is bounded for all $t>0$ by the extreme values of the boundary conditions and initial conditions, $\min\{T_-, \inf_\Omega\{T(t=0)\}\} \le T \le \max\{T_+, \sup_\Omega\{T(t=0)\}\}$, everywhere in Ω for $t>0$, even when $\phi \ne 0$.

Proof of Remark 2 follows immediately from the maximum principle applied to (3). Note that violation of the lower bound cited in Remark 2 would lead to a violation of the second law of thermodynamics; the upper bound follows similarly by symmetry arguments.

Remark 3: The long-time average of the heat flux through the upper and lower walls is equal, even when $\phi \ne 0$.

Proof of Remark 3 follows by integrating (3) over the domain Ω, which gives

$$\int_\Omega \left[\frac{\partial T}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} T = k\Delta T\right] d\mathbf{x}.$$

Integrating the second term by parts and integrating the third term directly, noting (1) and (4) and that $T_\pm = \pm 1$, leads to

$$\frac{d}{dt}\int_\Omega T\,d\mathbf{x} = \underbrace{\int_{\Gamma_+} k\frac{\partial T}{\partial y}\,d\mathbf{x} - \int_{\Gamma_-} k\frac{\partial T}{\partial y}\,d\mathbf{x}}_{\Psi_+ - \Psi_-}.$$

Taking the time average in the limit that $t \to \infty$, noting that $|\int_\Omega T\,d\mathbf{x}|$ is bounded (see Proposition A) leads to $\langle\Psi_+\rangle_\infty = \langle\Psi_-\rangle_\infty = \langle\bar{\Psi}\rangle_\infty$.

IV. Case B: The Boussinesq Approximation

We now consider the case with $H(\mathbf{u}) = 0$, as before, but now with $\mathbf{G}(T) = -\mathbf{g}\,T$, where $\mathbf{g}$ is proportional to the gravity vector. This is the Boussinesq approximation of the buoyancy force that arises due to temperature fluctuations in the flow.

Upon introducing the buoyancy effects into the model, the question of boundedness of $|T|_{L^2}$ becomes slightly more difficult to establish because the evolution of $\mathbf{u}$ is coupled to the evolution of T, so the boundedness of $|T|_{L^2}$ and $|\mathbf{u}|_{L^2}$ must be established at the same time.

Proposition B: In the case with $\mathbf{G}(T) = -\mathbf{g}\,T$ and $H(\mathbf{u}) = 0$, both $\int_\Omega T^2\,d\mathbf{x}$ and $|\int_\Omega T\,d\mathbf{x}|$ are uniformly bounded in t.

Proof: Decompose $\mathbf{u} = \mathbf{u}_\delta + \mathbf{u}_h$, where $\mathbf{u}_\delta \in H^1(\Omega)$ is a divergence-free "lifting function" specially constructed (see Hopf [15] and Temam [21]) to satisfy the boundary conditions on $\mathbf{u}$ with nonzero support focused within a neighborhood of the wall of width δ. The field $\mathbf{u}_h$ thus satisfies homogeneous boundary conditions. Inserting this decomposition into (2), multiplying by $\mathbf{u}_h$, and integrating over Ω gives

$$\int_\Omega \mathbf{u}_h\cdot\left[\frac{\partial\mathbf{u}_h}{\partial t} - \nu\Delta\mathbf{u}_h + \mathbf{u}_h\cdot\nabla\mathbf{u}_h + \nabla p + \mathbf{u}_h\cdot\nabla\mathbf{u}_\delta + \mathbf{u}_\delta\cdot\nabla\mathbf{u}_h = -\frac{\partial\mathbf{u}_\delta}{\partial t} + \nu\Delta\mathbf{u}_\delta - \mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta - \mathbf{i}P_x + \mathbf{G}(T)\right] d\mathbf{x}.$$

Noting that $\mathbf{u}_h$ and $\mathbf{u}_\delta$ are divergence-free and that $\mathbf{u}_h$ satisfies homogeneous boundary conditions, the third, fourth, and sixth terms vanish upon integration by parts. Integrating the second and fifth terms by parts, we may rewrite the result as

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}_h|^2_{L^2} + \nu|\nabla\mathbf{u}_h|^2_{L^2} = \int_\Omega \mathbf{u}_\delta\cdot[\mathbf{u}_h\cdot\nabla\mathbf{u}_h]\,d\mathbf{x} + \bar{f}(\mathbf{u}_\delta, P_x, T) \quad (10)$$

where

$$|\bar{f}(\mathbf{u}_\delta, P_x, T)| \le \left|\int_\Omega \mathbf{u}_h\cdot\frac{\partial\mathbf{u}_\delta}{\partial t}\,d\mathbf{x}\right| + \left|\nu\int_\Omega \mathbf{u}_h\cdot\Delta\mathbf{u}_\delta\,d\mathbf{x}\right| + \left|\int_\Omega \mathbf{u}_h\cdot[\mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta]\,d\mathbf{x}\right| + \left|\int_\Omega \mathbf{u}_h\cdot\mathbf{i}P_x\,d\mathbf{x}\right| + \left|\int_\Omega \mathbf{u}_h\cdot\mathbf{G}(T)\,d\mathbf{x}\right|. \quad (11)$$

It follows from Miranville and Wang [19, Lemma 1] and Cabral *et al.*[7, Lemma 5.1] that there exists a $\mathbf{u}_\delta \in H^1(\Omega)$ with $\mathbf{u}_\delta = -\phi_\pm\,\mathbf{n}$ on $\Gamma_\pm$ and $\nabla\cdot\mathbf{u}_\delta = 0$ such that

$$\left|\int_\Omega \mathbf{u}_\delta\cdot[\mathbf{u}_h\cdot\nabla\mathbf{u}_h]\,d\mathbf{x}\right| \le \delta|\nabla\mathbf{u}_h|^2_{L^2} \quad (12)$$

for any $\delta > 0$. Selecting $\delta = \nu/2$, taking the absolute value of both sides of (10) and applying (12) gives

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}_h|^2_{L^2} + \frac{\nu}{2}|\nabla\mathbf{u}_h|^2_{L^2} \le |\bar{f}(\mathbf{u}_\delta, P_x, T)|. \quad (13)$$

We now examine each term on the RHS of (11). Applying Cauchy–Schwarz, Poincaré, and then Young to the first term gives

$$\left|\int_\Omega \mathbf{u}_h\cdot\frac{\partial\mathbf{u}_\delta}{\partial t}\,d\mathbf{x}\right| \le |\mathbf{u}_h|_{L^2}\left|\frac{\partial\mathbf{u}_\delta}{\partial t}\right|_{L^2} \le \lambda_1|\nabla\mathbf{u}_h|_{L^2}\left|\frac{\partial\mathbf{u}_\delta}{\partial t}\right|_{L^2} \le \frac{\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2} + \frac{4\lambda_1^2}{\nu}\left|\frac{\partial\mathbf{u}_\delta}{\partial t}\right|^2_{L^2}. \quad (14a)$$

Applying integration by parts, Cauchy–Schwarz, and then Young to the second term of (11) gives

$$\left|\nu\int_\Omega \mathbf{u}_h\cdot\Delta\mathbf{u}_\delta\,d\mathbf{x}\right| \le \nu\left|\int_\Omega (\nabla\mathbf{u}_h)^*\cdot\nabla\mathbf{u}_\delta\,d\mathbf{x}\right| \le \nu|\nabla\mathbf{u}_h|_{L^2}|\nabla\mathbf{u}_\delta|_{L^2} \le \frac{\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2} + 4\nu\|\mathbf{u}_\delta\|^2_{H^1} \quad (14b)$$

where $()^*$ denotes transpose. Note that, by the third Sobolev inequality listed in Section II, when $\mathbf{u}\in V$ and $\mathbf{v},\mathbf{w}\in H^1(\Omega)$

$$\left|\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,d\mathbf{x}\right| \le C|\nabla\mathbf{u}|_{L^2}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{H^1}.$$

Applying this form of the Sobolev inequality then Young to the third term of (11) gives

$$\left|\int_\Omega \mathbf{u}_h\cdot[\mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta]\,d\mathbf{x}\right| \le C|\nabla\mathbf{u}_h|_{L^2}\|\mathbf{u}_\delta\|^2_{H^1} \le \frac{\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2} + \frac{4C^2}{\nu}\|\mathbf{u}_\delta\|^4_{H^1}. \quad (14c)$$

Applying Cauchy–Schwarz, Poincaré, and then Young to the fourth term of (11) gives

$$\begin{aligned}\left|\int_\Omega \mathbf{u}_h\cdot\mathbf{i}P_x\,d\mathbf{x}\right| &\le |P_x|\,\left|\int_\Omega \mathbf{u}_h\,d\mathbf{x}\right|\\ &\le |P_x|\,|\mathbf{u}_h|_{L^2}\le \lambda_1|P_x|\,|\nabla\mathbf{u}_h|_{L^2}\\ &\le \frac{\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2}+\frac{4\lambda_1^2}{\nu}P_x^2. \qquad (14\text{d})\end{aligned}$$

Applying Cauchy–Schwarz, Poincaré, and then Young to the fifth term of (11) gives

$$\begin{aligned}\left|\int_\Omega \mathbf{u}_h\cdot\mathbf{G}(T)\,d\mathbf{x}\right| &= \left|\int_\Omega \mathbf{u}_h\cdot\mathbf{g}T\,d\mathbf{x}\right|\\ &\le |\mathbf{g}|\,|\mathbf{u}_h|_{L^2}\,|T|_{L^2}\\ &\le \lambda_1^2 g\,|\nabla\mathbf{u}_h|_{L^2}\,|\nabla T|_{L^2}\\ &\le \frac{\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2}+\frac{4\lambda_1^4g^2}{\nu}|\nabla T|^2_{L^2} \qquad (14\text{e})\end{aligned}$$

where $g=|\mathbf{g}|$. We now decompose $T=T_\epsilon+T_h$, where T_ϵ is a lifting function similar to $\mathbf{u}_\delta$ that is specially constructed to satisfy the boundary conditions on T with nonzero support focused within a neighborhood of the wall of width ϵ. The field T_h thus satisfies homogeneous boundary conditions. Combining (13) with (11), noting (14a)–(14e) and the decomposition $T=T_\epsilon+T_h$, leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\mathbf{u}_h|^2_{L^2}+\frac{3\nu}{16}|\nabla\mathbf{u}_h|^2_{L^2}\\ &\quad\le \frac{4\lambda_1^2}{\nu}\left|\frac{\partial\mathbf{u}_\delta}{\partial t}\right|^2_{L^2}+4\nu\|\mathbf{u}_\delta\|^2_{H^1}\\ &\qquad+\frac{4C^2}{\nu}\|\mathbf{u}_\delta\|^4_{H^1}+\frac{4\lambda_1^2}{\nu}P_x^2\\ &\qquad+\frac{4\lambda_1^4g^2}{\nu}\left(\|T_\epsilon\|^2_{H^1}+|\nabla T_h|^2_{L^2}\right). \qquad (15)\end{aligned}$$

It follows from Cabral *et al.* [7, Lemma 5.2] that there exists a $T_\epsilon\in H^1(\Omega)$ with $T_\epsilon|_{\Gamma_-}=T_-=-1$ and $T_\epsilon|_{\Gamma_+}=T_+=1$ such that

$$\left|\int_\Omega T_\epsilon\cdot[\mathbf{u}\cdot\nabla T_h]\,d\mathbf{x}\right|\le\epsilon|\nabla\mathbf{u}|_{L^2}\,|\nabla T_h|_{L^2}. \qquad (16)$$

Now multiplying (3) by T_h, integrating over the domain Ω, and applying the decomposition $T=T_\epsilon+T_h$ gives

$$\int_\Omega T_h\left[\frac{\partial T_h}{\partial t}+\mathbf{u}\cdot\nabla T_\epsilon+\mathbf{u}\cdot\nabla T_h=k\Delta T_\epsilon+k\Delta T_h\right]d\mathbf{x}.$$

Integrating the second, third, fourth, and fifth terms by parts, noting (1) and $T_h|_{y=\pm1}=0$, then applying (16) and Cauchy–Schwarz gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|T_h|^2_{L^2}+k|\nabla T_h|^2_{L^2}\\ &\quad=\int_\Omega T_\epsilon\,\mathbf{u}\cdot\nabla T_h\,d\mathbf{x}-k\int_\Omega\nabla T_h\cdot\nabla T_\epsilon\,d\mathbf{x}\\ &\quad\le\epsilon\|\mathbf{u}\|_{H^1}\,|\nabla T_h|_{L^2}+k|\nabla T_h|_{L^2}\,|\nabla T_\epsilon|_{L^2}.\end{aligned}$$

Applying Young leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|T_h|^2_{L^2}+\frac{k}{2}|\nabla T_h|^2_{L^2}\\ &\quad\le\frac{\epsilon^2}{k}\left(\|\mathbf{u}_\delta\|^2_{H^1}+\left(1+\lambda_1^2\right)|\nabla\mathbf{u}_h|^2_{L^2}\right)+k|\nabla T_\epsilon|^2_{L^2}. \qquad (17)\end{aligned}$$

By (15) and (17), we have

$$\begin{aligned}&\frac{d}{dt}|\mathbf{u}_h|^2_{L^2}+\frac{3\nu}{8}|\nabla\mathbf{u}_h|^2_{L^2}\\ &\quad\le f_1(\mathbf{u}_\delta,T_\epsilon,P_x)+\frac{8\lambda_1^4g^2}{\nu}|\nabla T_h|^2_{L^2} \qquad (18\text{a})\\ &\frac{d}{dt}|T_h|^2_{L^2}+k|\nabla T_h|^2_{L^2}\\ &\quad\le f_2(\mathbf{u}_\delta,T_\epsilon)+\frac{2\left(1+\lambda_1^2\right)\epsilon^2}{k}|\nabla\mathbf{u}_h|^2_{L^2}. \qquad (18\text{b})\end{aligned}$$

Combining (18b) with $\alpha=k\nu/(16\lambda_1^4g^2)$ times (18a) gives

$$\begin{aligned}&\frac{d}{dt}\left(\alpha|\mathbf{u}_h|^2_{L^2}+|T_h|^2_{L^2}\right)+\frac{3\alpha\nu}{8}|\nabla\mathbf{u}_h|^2_{L^2}+k|\nabla T_h|^2_{L^2}\\ &\quad\le f_3(\mathbf{u}_\delta,T_\epsilon,P_x)+\frac{k}{2}|\nabla T_h|^2_{L^2}+\frac{2(1+\lambda_1^2)\epsilon^2}{k}|\nabla\mathbf{u}_h|^2_{L^2}.\end{aligned}$$

Selecting $\epsilon=\sqrt{\alpha\nu k/(16(1+\lambda_1^2))}$ and rearranging leads to

$$\begin{aligned}&\frac{d}{dt}\left(\alpha|\mathbf{u}_h|^2_{L^2}+|T_h|^2_{L^2}\right)+\frac{\alpha\nu}{4}|\nabla\mathbf{u}_h|^2_{L^2}\\ &\qquad+\frac{k}{2}|\nabla T_h|^2_{L^2}\le f_3(\mathbf{u}_\delta,T_\epsilon,P_x).\end{aligned}$$

Defining $\lambda=\min(\nu/4,\,k/2)$ and applying Poincaré gives

$$\begin{aligned}&\frac{d}{dt}\left(\alpha|\mathbf{u}_h|^2_{L^2}+|T_h|^2_{L^2}\right)+\lambda\left(\alpha|\mathbf{u}_h|^2_{L^2}+|T_h|^2_{L^2}\right)\\ &\qquad\le f_3(\mathbf{u}_\delta,T_\epsilon,P_x).\end{aligned}$$

Finally, multiplying by the integrating factor $e^{\lambda t'}$ and integrating over (0,t) gives

$$\begin{aligned}&\alpha|\mathbf{u}_h(t)|^2_{L^2}+|T_h(t)|^2_{L^2}\\ &\qquad\le\left(\alpha|\mathbf{u}_h(0)|^2_{L^2}+|T_h(0)|^2_{L^2}\right)e^{-\lambda t}+\frac{f_3}{\lambda}.\end{aligned}$$

Thus, both $|T(t)|_{L^2}$ and $|\mathbf{u}(t)|_{L^2}$ are uniformly bounded in t.□

Theorem B: The lowest sustainable net heat flux of an incompressible 2-D or 3-D channel flow [governed by (1)–(3)] under the Boussinesq approximation [$\mathbf{G}(T)=-\mathbf{g}T$ and $H(\mathbf{u})=0$], when controlled via a distribution of zero-net mass-flux blowing/suction over the channel walls (each with a constant temperature in x, z, and t) is exactly that of the laminar flow.

Proof of Theorem B, as well as the statement of proof of Corollary 1 and Remarks 2 and 3, follow exactly as for Case A in Section III, except for the slight modification of Corollary

1 that, if the $\mathbf{g}$ vector is not perpendicular to the walls, the laminar flow profile is a cubic instead of a parabola.

V. Case C: The Effect of Viscous Heating

We now consider the case with both $\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla\mathbf{u}) = c_2\sum_{i,j}((\partial u_i)/(\partial x_j) + (\partial u_j)/(\partial x_i))^2$. The latter term $H(\mathbf{u})$ accounts for the viscous heating in the incompressible flow (see, e.g., [4, Section 3.3]). This effect is significant in, for example, lubrication problems, in which the problem of minimizing heat flux is often an important engineering concern. Surprisingly, the two-way coupled equations, with both buoyancy [$\mathbf{G}(T)$] and viscous heating [$H(\mathbf{u})$] accounted for, have been studied relatively little in the literature on the Navier–Stokes equation. In [5] and [6], a generic heating term $\mathbf{H}$ was applied to the energy equation, albeit not specifically arising due to the viscocity of the fluid and the shear of the flow, as considered here.

Upon introducing viscous heating effects into the model, the question of boundedness of $|T(t)|_{L^2}$ becomes substantially more difficult to establish. In fact, it appears as if there is a mechanism that can, in some circumstances, lead to finite-time blow up of the two-way coupled equations. Stated precisely.

Conjecture: In the solution to the two-way coupled equations (1)–(3) with $\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla\mathbf{u})$, both $|T(t)|_{L^2}$ and $|\mathbf{u}(t)|_{L^2}$ can diverge to infinity in finite time.

Numerical evidence and phenomenological explanation of this mechanism for blowup in the finite-dimensional setting for a closely related problem of two coupled Burgers equations is presented in the Appendix. Of course, it is known that velocities and temperatures do not diverge in the actual fluid system. The possibility for divergence of the model (1)–(3) in this case is due to the breaking down of the approximations leading to these equations, such as the Boussinesq approximation of compressibility.

As a result of this mechanism for blowup in the two-way coupled equations (1)–(3), boundedness of $|T(t)|_{L^2}$ in this case can only be established under suitable "small data" assumptions. Further, once the boundedness of $|T(t)|_{L^2}$ is established (for conditions under which these assumptions are satisfied), the theorem that could be proved concerning the heat flux limitation in this case is both conservative and limited to the cold wall only, as shown below.

Proposition C: In the case with $\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla\mathbf{u})$, if the small data conditions A1–A6 (see below) are satisfied, the quantities $\int_\Omega T^2\,d\mathbf{x}$ and $|\int_\Omega T\,d\mathbf{x}|$ are bounded for all $t > 0$.

Proof: In a manner similar to the proof of Proposition B in Section IV, we decompose $\mathbf{u} = \mathbf{u}_\delta + \mathbf{u}_h$, where $\mathbf{u}_\delta$ is a specially constructed divergence-free "lifting function" with nonzero support focused within a neighborhood δ of the wall. Inserting this decomposition into (2), multiplying by $A\mathbf{u}_h$, and integrating over Ω gives

$$\int_\Omega A\mathbf{u}_h \cdot \left[\frac{\partial \mathbf{u}_h}{\partial t} - \nu\Delta\mathbf{u}_h + \mathbf{u}_h\cdot\nabla\mathbf{u}_h + \nabla p + \mathbf{u}_h\cdot\nabla\mathbf{u}_\delta + \mathbf{u}_\delta\cdot\nabla\mathbf{u}_h = \mathbf{f} - \mathbf{i}P_x - \mathbf{g}T\right] d\mathbf{x}$$

where $\mathbf{f} \triangleq -(\partial\mathbf{u}_\delta/\partial t) + \nu\Delta\mathbf{u}_\delta - \mathbf{u}_\delta\cdot\nabla\mathbf{u}_\delta$. Noting that $\mathbf{u}_h$ and $\mathbf{u}_\delta$ are divergence-free and that both $\mathbf{u}_h$ and $A\mathbf{u}_h$ satisfy homogeneous boundary conditions, the fourth term vanishes upon integration by parts, and the resulting expression may be written

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla\mathbf{u}_h|_{L^2}^2 + \nu|A\mathbf{u}_h|_{L^2}^2 &\le \int_\Omega |\mathbf{u}_h|\,|\nabla\mathbf{u}_h|\,|A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad + \int_\Omega |\mathbf{u}_h|\,|\nabla\mathbf{u}_\delta|\,|A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad + \int_\Omega |\mathbf{u}_\delta|\,|\nabla\mathbf{u}_h|\,|A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad + \int_\Omega |\mathbf{f}|\,|A\mathbf{u}_h|\,d\mathbf{x} + |P_x|\int_\Omega |A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad + g\int_\Omega |T|\,|A\mathbf{u}_h|\,d\mathbf{x} \end{aligned} \tag{19}$$

where $g = |\mathbf{g}|$. We now examine each term on the RHS of (19). Recalling (see, e.g., [11] and [21]) that the Sobolev inequalities listed in Section II imply

$$\begin{aligned}|\mathbf{u}|_{L^6} &\le C|\nabla\mathbf{u}|_{L^2} \\ |\nabla\mathbf{u}|_{L^3} &\le C|\nabla\mathbf{u}|_{L^2}^{1/2}\,|A\mathbf{u}|_{L^2}^{1/2} \quad \forall\,\mathbf{u}\in D(A)\end{aligned}$$

and applying Hölder, the above Sobolev inequalities, and then Young, the first term on the RHS of (19) may be written

$$\begin{aligned}\int_\Omega |\mathbf{u}_h|\,|\nabla\mathbf{u}_h|\,|A\mathbf{u}_h|\,d\mathbf{x} &\le |\mathbf{u}_h|_{L^6}\,|\nabla\mathbf{u}_h|_{L^3}\,|A\mathbf{u}_h|_{L^2} \\ &\le C|\nabla\mathbf{u}_h|_{L^2}^{3/2}\,|A\mathbf{u}_h|_{L^2}^{3/2} \\ &\le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_1}{\nu^3}|\nabla\mathbf{u}_h|_{L^2}^6.\end{aligned} \tag{20a}$$

Applying Hölder, Agmon and Poincaré, and then Young, the second term on the RHS of (19) may be written

$$\begin{aligned}&\int_\Omega |\mathbf{u}_h|\,|\nabla\mathbf{u}_\delta|\,|A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad\le |\mathbf{u}_h|_{L^\infty}\,|\nabla\mathbf{u}_\delta|_{L^2}\,|A\mathbf{u}_h|_{L^2} \\ &\quad\le C|\nabla\mathbf{u}_h|_{L^2}^{1/2}\,|\nabla\mathbf{u}_\delta|_{L^2}\,|A\mathbf{u}_h|_{L^2}^{3/2} \\ &\quad\le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_2}{\nu^3}|\nabla\mathbf{u}_h|_{L^2}^2\,|\nabla\mathbf{u}_\delta|_{L^2}^4.\end{aligned} \tag{20b}$$

Applying Hölder, Agmon, and then Young, the third term on the RHS of (19) may be written

$$\begin{aligned}&\int_\Omega |\mathbf{u}_\delta|\,|\nabla\mathbf{u}_h|\,|A\mathbf{u}_h|\,d\mathbf{x} \\ &\quad\le |\mathbf{u}_\delta|_{L^\infty}\,|\nabla\mathbf{u}_h|_{L^2}\,|A\mathbf{u}_h|_{L^2} \\ &\quad\le C\|\mathbf{u}_\delta\|_{H^1}^{1/2}\,\|\mathbf{u}_\delta\|_{H^2}^{1/2}\,|\nabla\mathbf{u}_h|_{L^2}\,|A\mathbf{u}_h|_{L^2} \\ &\quad\le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_3}{\nu}|\nabla\mathbf{u}_h|_{L^2}^2\,\|\mathbf{u}_\delta\|_{H^1}\,\|\mathbf{u}_\delta\|_{H^2}.\end{aligned} \tag{20c}$$

Applying Young, the fourth, fifth, and sixth terms on the RHS of (19) may be written

$$\int_\Omega |\mathbf{f}|\,|A\mathbf{u}_h|\,d\mathbf{x} \le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_4}{\nu}|\mathbf{f}|_{L^2}^2 \tag{20d}$$

$$|P_x|\int_\Omega |A\mathbf{u}_h|\,d\mathbf{x} \le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_4}{\nu}|P_x|^2 \tag{20e}$$

$$g\int_\Omega |T|\,|A\mathbf{u}_h|\,d\mathbf{x} \le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_5 g^2}{\nu}|T|_{L^2}^2. \tag{20f}$$

We now define

$$U^2 = \sup_{0\le t<\infty} \|\mathbf{u}_\delta\|_{H^1}^2\,\|\mathbf{u}_\delta\|_{H^2}^2$$
$$K^2 = \sup_{0\le t<\infty} \left[|\mathbf{f}|_{L^2}^2 + |P_x|^2\right].$$

Combining (20a)–(20f) into (19) and applying these definitions gives

$$\frac{d}{dt}|\nabla\mathbf{u}_h|_{L^2}^2 + \nu|A\mathbf{u}_h|_{L^2}^2 \le R$$

where

$$\begin{aligned} R \triangleq{} & \frac{C_1|\nabla\mathbf{u}_h|_{L^2}^4}{\nu^3}|\nabla\mathbf{u}_h|_{L^2}^2 \\ &+ \frac{C_2U^2}{\nu^3}|\nabla\mathbf{u}_h|_{L^2}^2 + \frac{C_4}{\nu}K^2 \\ &+ \frac{C_5g^2}{\nu}|T|_{L^2}^2 + \frac{C_3U}{\nu}|\nabla\mathbf{u}_h|_{L^2}^2. \end{aligned}$$

Applying Poincaré, we may rewrite this as

$$\frac{d}{dt}|\nabla\mathbf{u}_h|_{L^2}^2 + \frac{\nu}{2\lambda_1}|\nabla\mathbf{u}_h|_{L^2}^2 + \frac{\nu}{2}|A\mathbf{u}_h|_{L^2}^2 \le R. \tag{21}$$

We now make the following assumptions:

$$\frac{C_1|\nabla\mathbf{u}_h(0)|_{L^2}^4}{\nu^3} + \frac{C_1\|\mathbf{u}_\delta(0)\|_{H^1}^4}{\nu^3} \le \frac{\nu}{6\lambda_1} \tag{A1}$$

$$\frac{C_2U^2}{\nu^3} + \frac{C_3U}{\nu} \le \frac{\nu}{6\lambda_1} \tag{A2}$$

$$\frac{C_5\,g^2\,\lambda_1^2}{\nu} \le \frac{k}{4}. \tag{A3}$$

Additionally, we select some $t_1 > 0$ such that

$$\frac{C_1|\nabla\mathbf{u}_h|_{L^2}^4}{\nu^3} \le \frac{\nu}{3\lambda_1} \text{ for all } t\in[0,t_1). \tag{22}$$

These assumptions, together with the decomposition $T = \bar{T} + T'$ and Poincaré, allow us to rewrite (21) as

$$\frac{d}{dt}|\nabla\mathbf{u}_h|_{L^2}^2 + \frac{\nu}{2}|A\mathbf{u}_h|_{L^2}^2 \le M + \frac{k}{4}|\nabla T'|_{L^2}^2 \text{ for } t\in[0,t_1) \tag{23}$$

where

$$M \triangleq \frac{C_4}{\nu}K^2 + \frac{C_5g^2}{\nu}|\bar{T}|_{L^2}^2 + \frac{\lambda_1^2}{2k}|\mathbf{u}_\delta|_{L^2}^2.$$

Now decomposing $T = \bar{T} + T'$, where $\bar{T} = y$, then multiplying (3) by T' and integrating over the domain Ω gives

$$\int_\Omega T'\left[\frac{\partial T'}{\partial t} + \mathbf{u}\cdot\nabla\bar{T} + \mathbf{u}\cdot\nabla T' = k\Delta T' + H(\mathbf{u})\right]d\mathbf{x}. \tag{24}$$

By the same steps as those leading to (25), the definition of $H(u)$, and Poincaré and Hölder, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|T'|_{L^2}^2 + \frac{k}{2}|\nabla T'|_{L^2}^2 \\ &\quad\le \frac{\lambda_1^2}{2k}|\mathbf{u}|_{L^2}^2 + \int_\Omega H(\mathbf{u})T'\,d\mathbf{x} \\ &\quad\le \frac{\lambda_1^2}{2k}|\mathbf{u}|_{L^2}^2 + C\int_\Omega |\nabla\mathbf{u}|^2T'\,d\mathbf{x} \\ &\quad\le \frac{\lambda_1^2}{2k}|\mathbf{u}_\delta|_{L^2}^2 + \frac{\lambda_1^6}{2k}|A\mathbf{u}_h|_{L^2}^2 \\ &\qquad + C_6|\nabla\mathbf{u}|_{L^3}^2\,|T'|_{L^3}. \end{aligned} \tag{25}$$

The RHS in the inequality above can be bounded by

$$\frac{\lambda_1^6}{2k}|A\mathbf{u}_h|_{L^2}^2 + C_6|\nabla\mathbf{u}_h|_{L^3}^2\,|T'|_{L^3} + \frac{\lambda_1^6}{2k}|A\mathbf{u}_\delta|_{L^2}^2 + C_6|\nabla\mathbf{u}_\delta|_{L^3}^2\,|T'|_{L^3}.$$

Recalling (see, e.g., [11] and [21]) that the Sobolev inequalities listed in Section II imply

$$\nabla\mathbf{u}_h|_{L^3} \le C\nabla\mathbf{u}_h|_{L^2}^{1/2}\,|A\mathbf{u}_h|_{L^2}^{1/2}$$
$$|T'|_{L^3} \le C|T'|_{L^2}^{1/2}\,|\nabla T'|_{L^2}^{1/2}$$

and applying the above Sobolev inequalities, Young, and then Poincaré, we have

$$\begin{aligned} &C_6|\nabla\mathbf{u}_h|_{L^3}^2\,|T'|_{L^3} \\ &\quad\le C|\nabla\mathbf{u}_h|_{L^2}\,|A\mathbf{u}_h|_{L^2}\,|T'|_{L^2}^{1/2}\,|\nabla T'|_{L^2}^{1/2} \\ &\quad\le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C}{\nu}|\nabla\mathbf{u}_h|_{L^2}^2\,|T'|_{L^2}\,|\nabla T'|_{L^2} \\ &\quad\le \frac{\nu}{16}|A\mathbf{u}_h|_{L^2}^2 + \frac{C_7\lambda_1}{\nu}|\nabla\mathbf{u}_h|_{L^2}^2\,|\nabla T'|_{L^2}^2. \end{aligned}$$

Making the additional assumptions

$$\frac{\lambda_1^6}{k} \le \frac{\nu}{8} \tag{A4}$$

$$\frac{C_7\lambda_1}{\nu}\sqrt{\frac{\nu^4}{3C_1\lambda_1}} \le \frac{k}{4} \tag{A5}$$

and applying (22), we may rewrite (25) as

$$\frac{d}{dt}|T'|_{L^2}^2 + \frac{k}{2}|\nabla T'|_{L^2}^2 \le \frac{\nu}{4}|A\mathbf{u}_h|_{L^2}^2 + \frac{\lambda_1^2}{2k}|\mathbf{u}_\delta|_{L^2}^2 \text{ for } t\in[0,t_1). \tag{26}$$

Adding (23) and (26) gives

$$\frac{d}{dt}\left[|\nabla \mathbf{u}_h|_{L^2}^2 + |T'|_{L^2}^2\right] + \frac{\nu}{4}|A\mathbf{u}_h|_{L^2}^2 + \frac{k}{4}|\nabla T'|_{L^2}^2 \leq M \text{ for } t \in [0, t_1).$$

Defining $\alpha = \min(\nu, k)\lambda_1^2/4$ and assuming

$$\frac{M}{\alpha} < \nu^2\sqrt{\frac{C_1}{6\lambda_1}} \tag{A6}$$

applying Poincaré gives

$$\frac{d}{dt}\left[|\nabla \mathbf{u}_h|_{L^2}^2 + |T'|_{L^2}^2\right] + \alpha\left[|\nabla \mathbf{u}_h|_{L^2}^2 + |T'|_{L^2}^2\right] \leq M \text{ for } t \in [0, t_1).$$

Thus, applying Gronwall's lemma

$$|\nabla \mathbf{u}_h|_{L^2}^2 + |T'|_{L^2}^2 \leq \left[|\nabla \mathbf{u}_h(0)|_{L^2}^2 + |T'(0)|_{L^2}^2\right] e^{-\alpha t} + \frac{M}{\alpha}, \text{ for } t \in [0, t_1)$$

from which we see that (22) is true for all $t_1 > 0$, and therefore the estimates above are valid for all time. □

Remark 5: In Proposition C, Assumptions A1–A6 are all satisfied if λ_1 is sufficiently small, which is true if domain is sufficiently thin.

Remark 6: In the case (not studied here) of (1)–(3) with $\mathbf{G}(T) = 0$ and $H(\mathbf{u}) = -c_1(\boldsymbol{\tau} : \nabla \mathbf{u})$, existence of renormalized solutions for all $t > 0$ has been established without a small data assumption by Lions [18] using a proof based on conservation of energy and the method of renormalization. The proof presented here treats the case in which there is a stronger coupling between the equations for $\mathbf{u}$ and T.

Theorem C: The lowest sustainable heat flux through the cold wall $[\Gamma_-]$] of an incompressible 2-D or 3-D channel flow [governed by (1)–(3)] with both buoyancy and viscous heating effects accounted for [$\mathbf{G}(T) = -\mathbf{g}T$ and $H(\mathbf{u}) = -c_1(\tau : \boldsymbol{\nabla}\mathbf{u})$], when controlled via a distribution of zero-net mass-flux blowing/suction over the channel walls (each with a constant temperature in x, z, and t) is bounded from below by that of the laminar flow when no viscous heating is present.

Proof: Decompose T into two components, $T = T' + T''$. Let T' be defined as the solution of

$$\frac{\partial T'}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T' = k\Delta T' + H(\mathbf{u}) \tag{27}$$

with $T'|_{\Gamma_\pm} = 0$ and $T'(t = 0) = 0$. Since the forcing on this system is nonnegative, $H(\mathbf{u}) \geq 0$, it follows by the minimum principle that $T' \geq 0$ everywhere in Ω. Thus

$$\left\langle \int_{\Gamma_-} k\frac{\partial T'}{\partial y}\, d\mathbf{x} \right\rangle_\infty \geq 0. \tag{28}$$

A lower bound (equality in the above expression) is achieved by the case with $H(\mathbf{u}) = 0$, from which it follows from (27) that $T' = 0$ everywhere in Ω; note that, for the actual problem of interest, with $H(\mathbf{u}) = -c_1(\tau : \boldsymbol{\nabla}\mathbf{u})$ and $\mathbf{u} \neq 0$, this lower bound is conservative.

Let T'' be defined as the solution of

$$\frac{\partial T''}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T'' = k\Delta T'' \tag{29}$$

with initial conditions $T''(t = 0) = T_0$, where $T_0 \in L^2(\Omega)$, and boundary conditions $T''|_{\Gamma_+} = T_+ = 1, T''|_{\Gamma_-} = T_- = -1$. By the same derivation as in the proof of Remark 3 of Section III, it follows that

$$\left\langle \int_{\Gamma_+} k\frac{\partial T''}{\partial y}\, d\mathbf{x} \right\rangle_\infty = \left\langle \int_{\Gamma_-} k\frac{\partial T''}{\partial y}\, d\mathbf{x} \right\rangle_\infty.$$

Thus, by the same derivation as in the proof of Theorem A of Section III, it follows that

$$\left\langle \int_{\Gamma_-} k\frac{\partial T''}{\partial y}\, d\mathbf{x} \right\rangle_\infty = kL_xL_z + \frac{k}{2}\left\langle |\boldsymbol{\nabla}(T'' - y)|_{L^2}^2 \right\rangle_\infty. \tag{30}$$

Thus, the minimum value of $\langle \int_{\Gamma_-} k(\partial T'')/(\partial y)\, d\mathbf{x}\rangle_\infty$ is achieved when the state $T'' = y$ is exponentially stabilized.

Finally, by adding (27) and (29), note that $T = T' + T''$ satisfies (3), in addition to satisfying the necessary initial and boundary conditions on T. By (28) and (30), the sum

$$\langle \Psi_- \rangle_\infty = \left\langle \int_{\Gamma_-} k\frac{\partial T}{\partial y}\, d\mathbf{x} \right\rangle_\infty = \left\langle \int_{\Gamma_-} k\frac{\partial T'}{\partial y}\, d\mathbf{x} \right\rangle_\infty + \left\langle \int_{\Gamma_-} k\frac{\partial T''}{\partial y}\, d\mathbf{x} \right\rangle_\infty$$

is bounded from below by the laminar temperature profile in the case when no viscous heating is present (that is, by $T = T' + T'' = 0 + y = y$). □

VI. Discussion

If the laminar flow is globally exponentially stable [that is, if the Reynolds number characterizing the stability of (2) and the Rayleigh number characterizing stability of (3) are sufficiently small], the flow will eventually converge exponentially quickly to the laminar state with no control effort applied (that is, with $\phi = 0$). This paper proves that such relaminarization achieves the minimum sustainable net heat flux in both 2-D and 3-D channel flows, even when both buoyancy effects (via the Boussinesq approximation) and viscous heating effects are accounted for.

If the laminar flow is unstable (that is, if the Reynolds number or Rayleigh number is large), control forcing is necessary to stabilize the laminar state in order to achieve the minimum sustainable net heat flux. This paper establishes that the objectives of minimizing heat flux and stabilizing the laminar state are equivalent objectives in the control design for unstable systems of this sort. (In other words, in order to minimize net heat flux, one should attempt to drive the flow to the laminar state rather than to some peculiar unsteady motion.) Under what conditions

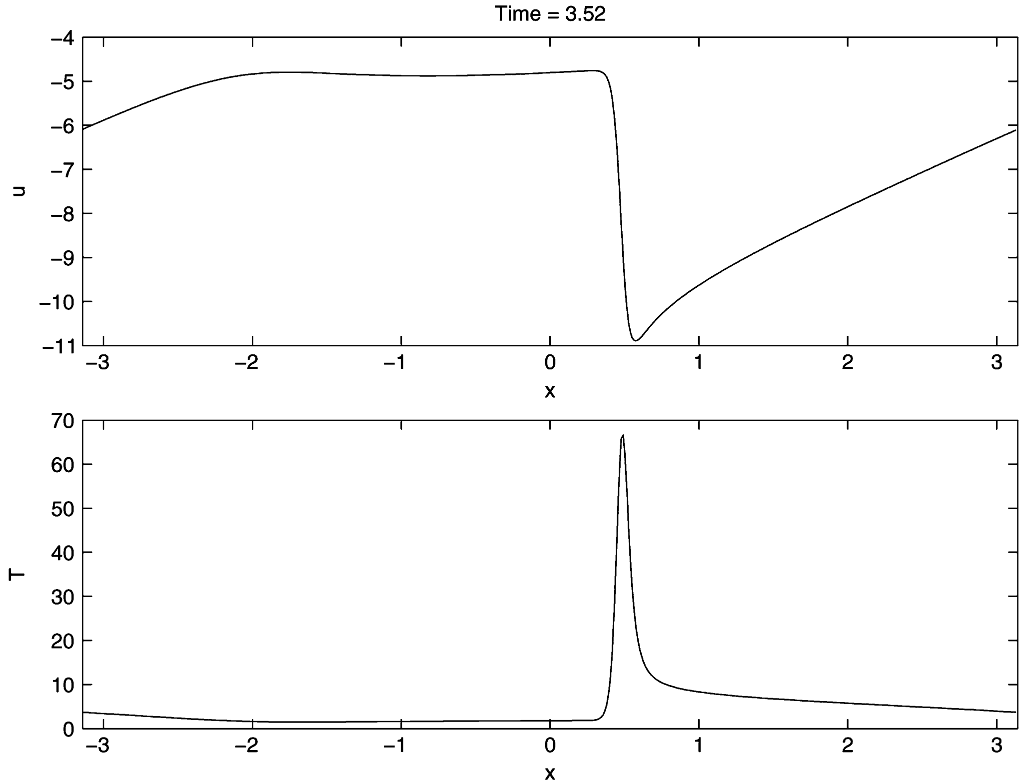

Fig. 1. Simulation of the two-way coupled Burgers system (31a) and (31b).

such stabilization is actually possible, and with what specific control algorithm, are separate questions that are not addressed here. However, the knowledge that two different control objectives are equivalent is in fact quite useful from the perspective of control design, as it allows one to pose alternative yet equivalent control problems, some of which might be easier to solve than others.

Phenomenological justification of the results presented in Sections III and IV (in the cases with $H(\mathbf{u}) = 0$) is given by the following argument: by Remark 2, if $|T_0| \leq 1$ (e.g., if $T(t=0) = y$), then $-1 \leq T \leq 1$ everywhere in Ω for $t > 0$. Near the lower wall, flowfield unsteadiness thus inevitably transports warmed fluid (with $T > -1$) towards the cold wall (at $T = -1$) and cooled fluid (with $T \approx -1$) away from the cold wall, thereby accelerating the effect of diffusion. [Similarly, near the upper wall, flowfield unsteadiness inevitably transports cooled fluid (with $T < 1$) towards the hot wall (at $T = 1$) and hot fluid (with $T \approx 1$) away from the hot wall.] Thus, this proof coincides with the (perhaps strong) physical intuition that flowfield unsteadiness necessarily accelerates net heat transport in the direction of diffusion.

APPENDIX

The proof of existence of the two-way coupled system in Section V was stated under a small data assumption. Remark 4 noted that this two-way coupled system can actually blow up. The mechanism for this blowup can be investigated numerically by considering a one-dimensional analog of the system (1)–(3); that is, consider now the two-way coupled Burgers system

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} - gT \tag{31a}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2. \tag{31b}$$

A typical simulation result (with $\nu = k = 0.1, g = 1, u(0) = \sin(x), T(0) = \cos(x)$, and periodic boundary conditions) is shown in Fig. 1. A mixed RKW3/CN timestepping algorithm [2] is used in the simulation, with spectral differentiation in space using 512 grid points and a fixed timestep $dt = 0.002$.

Note that a steep shear layer forms in the velocity field, and a sharp peak forms in the temperature field. As the simulation proceeds, the shear in the velocity field creates an energy source in the temperature equation, further sharpening the temperature peak. Simultaneously, the peak in the temperature field creates a nonuniform momentum source which further steepens the shear layer in the velocity field. Thus, the two phenomena reinforce each other, and the L^2 and H^1 norms of the system escape to infinity in finite time. Similar simulation results are also obtained when Dirichlet or Neumann boundary conditions are used, so long as the domain is taken to be sufficiently large that the system diverges before the peak in the temperature field convects to the boundary of the domain.

REFERENCES

[1] T. R. Bewley, "Flow control: New challenges for a new renaissance," *Prog. Aerosp. Sci.*, vol. 37, pp. 21–58, Jan. 2001.

[2] T. R. Bewley, P. Moin, and R. Temam, "DNS-based predictive control of turbulence: An optimal benchmark for feedback algorithms," *J. Fluid Mech.*, vol. 447, pp. 179–225, 2001.

[3] T. R. Bewley, R. Temam, and M. Ziane, "A general framework for robust control in fluid mechanics," *Phys. D.*, vol. 138, pp. 360–392, Apr. 2000.

[4] R. B. Bird, W. E. Stewart, and E. N. Lightfoot*, Transport Phenomena*. New York: Wiley, 1960.

[5] J. Boland and W. Layton, "Error analysis for finite element methods for steady natural convection problems," *Numer. Functional Anal. Optim.*, vol. 11, no. 5, pp. 449–483, May 1990.

[6] J. Boland and W. Layton, "Error analysis for finite element methods for steady natural convection problems," *Numer. Functional Anal. Optim.*, vol. 11, no. 5, pp. 449–483, May 1990.

[7] M. Cabral, R. Rosa, and R. Temam, "Existence and dimension of the attractor for the Bénard problem on channel-like domains," *Discrete Continuous Dyn. Syst.*, to be published.

[8] S. S. Collis, R. D. Joslin, A. Seifert, and V. Theofilis, "Issues in active flow control: Theory, control, simulation, and experiment," *Prog. Aerosp. Sci.*, vol. 40, pp. 237–289, 2004.

[9] P. Constantin and C. R. Doering, "Variational bounds on energy dissipation in incompressible flows—I: Shear flow," *Phys. Rev. E.*, vol. 49, pp. 4087–4099, May 1994.

[10] P. Constantin and C. R. Doering, "Variational bounds on energy dissipation in incompressible flows—II. Channel flow," *Phys. Rev. E.*, vol. 51, pp. 3192–3198, Apr. 1995.

[11] P. Constantin and C. Foias*, Navier-Stokes Equations*. Chicago, IL: Univ. of Chicago Press, 1988.

[12] J. S. Freudenberg and D. P. Looze, "Right half plane poles and zeros and design tradeoffs in feedback systems," *IEEE Trans. Autom. Control*, vol. AC-30, pp. 555–565, Jun. 1985.

[13] M. D. Gunzburger*, Perspectives in Flow Control and Optimization*. Philadelphia, PA: SIAM, 2002.

[14] M. Högberg, T. R. Bewley, and D. S. Henningson, "Linear feedback control and estimation of transition in plane channel flow," *J. Fluid Mech.*, vol. 481, pp. 149–175, 2003.

[15] E. Hopf, "On nonlinear partial differential equations," in *Lecture Series of the Symposium on Partial Differential Equations*, Berkeley, CA, 1955, pp. 1–29.

[16] J. Kim, "Control of turbulent boundary layers," *Phys. Fluids*, vol. 15, pp. 1093–1105, 2003.

[17] J. Kim and T. Bewley, "A linear systems approach to flow control," *Annu. Rev. Fluid Mech.*, vol. 39, Jan. 2007.

[18] P.-L. Lions, "Mathematical topics in fluid mechanics," in *Incompressible Models*. Oxford, U.K.: Oxford Univ. Press, 1996, vol. 1.

[19] A. Miranville and X. Wang, "Attractors for nonautonomous nonhomogeneous Navier-Stokes equations," *Nonlinearity*, vol. 10, pp. 1047–1061, Sep. 1997.

[20] M. Seron, J. Braslavsky, and G. Goodwin*, Fundamental Limitations in Filtering and Control*. Berlin, Germany: Springer, 1997.

[21] R. Temam*, Navier-Stokes Equations: Theory and Numerical Analysis*. Amsterdam, The Netherlands: North Holland, 1984.

[22] R. Temam*, Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Berlin, Germany: Springer, 1988.

Thomas R. Bewley, photograph and biography not available at the time of publication.

Mohammed Ziane, photograph and biography not available at the time of publication.